A Genetic Context for Understanding the Trigonometric Functions Episode 6: Regiomontanus and the Beginnings of Modern Trigonometry

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Trigonometry is concerned with the measurements of angles about a central point (or of arcs of circles centered at that point) and quantities, geometrical and otherwise, which depend on the sizes of such angles (or the lengths of the corresponding arcs). It is one of those subjects that has become a standard part of the toolbox of every scientist and applied mathematician. Today an introduction to trigonometry is normally part of the mathematical preparation for the study of calculus and other forms of mathematical analysis, as the trigonometric functions make common appearances in applications of mathematics to the sciences, wherever the mathematical description of cyclical phenomena is needed. This project is one of a series of curricular units that tell some of the story of where and how the central ideas of this subject first emerged, in an attempt to provide context for the study of this important mathematical theory. Readers who work through the entire collection of units will encounter six milestones in the history of the development of trigonometry. In this unit, we read selections from a systematic work on trigonometry written in the fifteenth century in central Europe to see how the subject began to be separated from its practical astronomical roots to become a purely geometric study of plane triangles and the measures of their angles and sides. Connections will be made between this work and the modern practice of defining trigonometric quantities as ratios of side lengths in a right triangle.

1 Regiomontanus' *On Triangles of Every Kind*

Johannes Müller von Königsberg (1436–1476) was born just outside the town of Königsberg (located in Lower Franconia in central Germany) and studied as a young man at the universities in Leipzig and Vienna. Because he wrote in Latin, as did most European scholars of the day, he signed his works with the Latinized form of his name: *Ioannes de Monte Regio*[1] A later writer would refer to him as Regiomontanus, the name by which he has been generally known since [Katz, 1998, Van Brummelen, 2009].

While in Vienna, Regiomontanus became the student of Georg Peuerbach (1421–1461), a mathematician and astronomer who gained prominence as court astrologer for King Ladislas V of Bohemia and Hungary. The two worked closely with each other for the rest of Peuerbach's short life. Through connections with Peuerbach, Regiomontanus was given the opportunity to travel across Europe in

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[1] *Monte Regio* is the Latin form of *Königsberg*, which in English means "King's Mountain."

the company of Cardinal Johannes Bessarion (1403–1472), legate to the pope and himself a great scholar; this position allowed Regiomontanus unique access to the most prominent thinkers of his day and to some of the best libraries on the European continent.

After Peuerbach's death, Regiomontanus completed a critical edition of an important Greek astronomical work, Ptolemy's *Almagest*,[2] a project that Peuerbach had begun and that was left for Regiomontanus to complete after the older man's death. In the process he became acquainted with works by Arabic scholars in trigonometry and decided that a systematic account of that subject in support of the needs of astronomers ought to be made. To that end he wrote *De triangulis omnimodis* (*On Triangles of Every Kind*), a work in five Books, in 1464.[3]

In this project, we will consider some selections from Book I of *On Triangles*. Before we begin reading from that work, however, let us first take a brief look at the state of trigonometrical knowledge of Regiomontanus and his contemporaries at the time of the European Renaissance.

2 Trigonometry at the time of Regiomontanus

From the dawn of recorded history, there are accounts of people trying to make sense of what they saw when they looked up to the sky. These accounts described the awesome beauty and amazing regularity of the heavenly objects they beheld there, and they often located in the heavens explanations for what they could not otherwise explain about their lives on earth. The first science was astronomy and mathematics served this science in key ways.

The ancients believed that the sky was a great sphere and that earth sat at the center of this sphere with the Sun, the Moon, and the other visible planets[4] moving in circular orbits about earthbound observers. Consequently, the geometry of circle and sphere was central[5] to descriptions of their behaviors. It was in these theoretical and practical developments that trigonometry was born.

To mathematician-astronomers of the Old Babylonian era (second millennium BCE) we attribute the practice used to this day of dividing the circle into 360 parts ($= 6 \times 60$) in accordance with the *sexagesimal*, or base 60, numeration of their culture to measure portions of the arcs of circles.[6]

To the ancient Greeks we are indebted for a theoretical science of circle geometry that was employed to provide both geometric and quantitative procedures for locating, following, and predicting the motions of objects in the sky. A characteristic development of this sort was the *table of chords* in a circle. The diagram below displays a portion of a circle centered at O with central angle θ and

[2]Claudius Ptolemy (fl. ca. 250 CE) was a highly influential Greek mathematician whose work, *The Mathematical Collection* came to be called *The Great Collection*, or in Arabic *Kitāb al-majistī*, by the later scholars of that era and culture who studied and used it. The European scholars who produced Latin translations of this edition transliterated the Arabic title as *Almagest*.

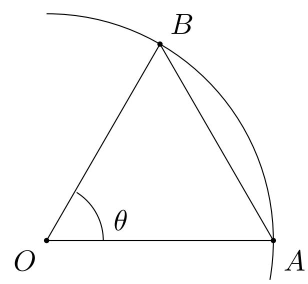
[3]At its completion in 1464, *On Triangles* only existed in handwritten manuscript editions; printing was only just introduced as a means for spreading the work of authors and not yet an option to Regiomontanus. (The first printed book in Europe was—famously—the Gutenberg Bible; it was created in 1455.) The first printed copy of *On Triangles* was produced posthumously, in 1533, and a second edition came out in 1561. The English translation from which the excerpts we use in this project were taken was prepared using the 1533 edition and was published by Barnabas Hughes, O.F.M., in 1967 [Hughes, 1967].

[4]In the ancient world, any object in the sky that moved independently of the background stars (including the Sun and Moon) was identified as a planet (the Greek word *planetos* meant *wanderer*).

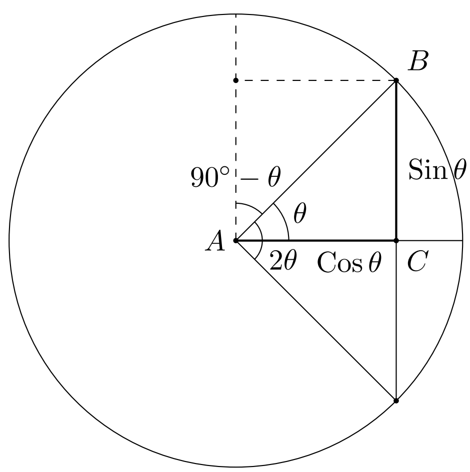
[5]Pun intended!

[6]For more on Babylonian sexagesimal numeration and its enduring legacy in the measurement of circular arcs and angles, see the project "Babylonian Astronomy and Sexagesimal Numeration" from the *Convergence* article "Teaching and Learning the Trigonometric Functions through Their Origins."

associated circular arc $\overset{\frown}{AB}$ and linear chord $\overline{AB}$. In a two-column chord table, one column systematically listed the lengths of arcs along a circle ($\overset{\frown}{AB}$) in multiples of some small fraction of the full circumference while the other gave the lengths of the corresponding chords.[7] Note that the length of the chord, denoted $\mathrm{Crd}\,\theta$, is always shorter than the length of the arc, since a straight line must be the shortest distance between the two points. Both these lengths in turn depend on the radius of the circle.



In later centuries and in other parts of the world, the spread of Greek science enticed other scholars to further developments in trigonometry. In sixth-century (CE) India, Hindu astronomers realized that astronomical tables would be more efficiently employed if instead of chords of arcs, they tabulated the measures of half-chords. As in the following diagram, if one knows the measure of the angle θ, then half the chord corresponding to 2θ is the length of the side of a right triangle that is opposite the angle θ. This half-chord was called the *Sine* of the arc, and was abbreviated $\mathrm{Sin}\,\theta$.



Thus began the custom in trigonometry of focusing on the Sine (or sine) of an arc as opposed to its chord.[8] Note that when θ measures a given number of degrees, the length of $\mathrm{Sin}\,\theta$ will be

[7] The earliest table of chords is thought to have been constructed by Hipparchus of Rhodes in the second century BCE. Such tables were used to help solve important astronomical problems like using the length of the shadow cast by a vertical stick to tell how high in the sky the Sun was, and therefore to tell the time of day. Problems of this type, together with others like when to expect a given star or planet to rise, or when an eclipse might occur, as well as solutions to these problems, can be found in Ptolemy's *Almagest*. This helps to explain why the *Almagest* became the definitive authority for mathematical astronomy for many centuries after its writing in the third century CE. You can learn more about Greek contributions to the development of trigonometry work in the projects "Hipparchus' Table of Chords" and "Ptolemy Finds High Noon in Chords of Circles" from the *Convergence* article "Teaching and Learning the Trigonometric Functions through Their Origins."

[8] For more the origin of the sine and the fascinating way in which Hindu astronomical learning was transmitted via Sanskrit verse from generation to generation, see the project "Varāhamihira and the Poetry of Sines" from the *Convergence* article "Teaching and Learning the Trigonometric Functions through Their Origins."

proportionally larger in a circle with a larger radius; we capitalize the letter 'S' in Sine to remind us of this dependence on the size of the circle. If however we assume that the circle has unit radius, we conventionally use the lowercase form $\sin\theta$. This accords with modern practice of associating lowercase sines with the unit circle. The Cosine (or cosine in a circle of unit radius) of an arc θ, abbreviated $\operatorname{Cos}\theta$ (or $\cos\theta$) was the Sine (or sine) of the complementary arc $90^\circ - \theta$, hence it represented the other side of the right triangle (the side AC in the figure above).

By the eleventh century, Islamic mathematicians had extended their trigonometric tables to include the measures of other segments related to arcs in a circle; we will consider these briefly later in this project. Beginning in the twelfth century, the works of these Arabic astronomers, and especially translations of these works into Latin, had begun to influence European astronomers in the new Western universities of Oxford and Paris. In central Europe, however, far from these universities scholars like Regiomontanus were not yet familiar with the latest advances. They relied instead on the authority of Ptolemy's *Almagest* and on other Arabic sources that used only sine tables and certain theorems about relationships involving them, about which we will say more below. This is why, in the note "To Readers" at the beginning of *On Triangles*, Regiomontanus offered the following advice:

> [N]o one can bypass the science of triangles and reach a satisfying knowledge of the stars ... You, who wish to study great and wonderful things, who wonder about the movement of the stars, must read these theorems about triangles. Knowing these ideas will open the door to all astronomy and to certain geometric problems.

3 What's your angle?

In the light of the story we have told above about the development of trigonometry out of problems in mathematical astronomy, there is one surprising fact about Regiomontanus' *On Triangles*: despite the remarks made above by its author, readers of this work will not find a single problem in astronomy solved here! Instead, the problems here are of the form: determine the unknown sides and angles of a triangle given values of certain of its sides and angles. Indeed, *On Triangles* is a systematic treatment of the *solving of triangles* (in the plane and on the sphere) as a purely mathematical problem. Applications to astronomy have been set aside. Apparently, Regiomontanus was interested here only in preparing the reader to hone the proper geometric tools needed for the study of astronomy. Because of this, the organizational style of *On Triangles* is very much in the classical geometric style, like Euclid's *Elements*, as opposed to astronomical works like Ptolemy's *Almagest*.

For example, Book I of *On Triangles* opens in the same way as Euclid's *Elements*, with definitions of key terms like *side* and *angle* in a triangle, *arc* of a circle, and *right Sine*, or simply *Sine*, of arc in a circle.[9] It should be noted that Regiomontanus did not use the uppercase/lowercase notational convention of restricting the term sine for circles of unit radius versus Sine for circles with other

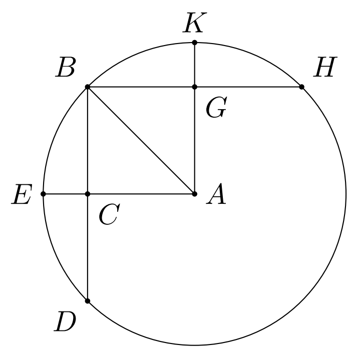
[9] The word *right* is not meant to signify a measure of 90°, in the way that right angle does; it simply means that the Sine is the line segment perpendicular to the radius at the initial point of the arc.

radius measures. Therefore, whenever the term *sine* appears in an excerpt below, we have changed it to Sine to make it clear that the radius of the underlying circle is not specified to equal 1.

After a number of elementary theorems regarding ratios of line segments and areas of rectangles, Regiomontanus finally turned to triangles in his Theorem 20.

Theorem 20. In every right triangle, one of whose acute vertices becomes the center of a circle and whose [hypotenuse is] its radius, the side subtending this acute angle is the right Sine of the arc adjacent to that [side and] opposite the given angle, and the third side of the triangle is equal to the Sine of the complement of the arc.

If a right triangle ABC is given with C the right angle and A an acute angle, around the vertex of which a circle BED is described, with the hypotenuse—that is, the side opposite the largest angle—as radius, and if side AC is extended sufficiently to meet the circumference of the circle at point E, then side BC opposite angle BAC is the Sine of arc BE subtending the given angle, and furthermore the third side AC is equal to the right Sine of the complement of arc BE.



Task 1 Regiomontanus presented Theorem 20 in two paragraphs, the first of which stated the assertion in general terms, the second of which referred to a particular labeled diagram to explain the general assertion. For each of the phrases listed below from the first paragraph, identify the element of the labeled diagram, as explicated in the second paragraph, that refers to the same object.

(a) "right triangle" (Which angle of the triangle is the right angle?)

(b) "one of whose acute vertices becomes the center of a circle"

(c) "whose [hypotenuse is] its radius"

(d) "the side subtending this acute angle"

(e) "the right Sine of the arc"

(f) "that [side]" (to which "the right Sine of the arc" is "adjacent")

(g) "the given angle" (to which "the right Sine of the arc" is "opposite")

(h) "the third side of the triangle ... equal to the Sine of the complement of the arc"

(i) Which angle is the angle whose Cosine, i.e., "the Sine of the complement," is the object described in part (h) above?

Notice that Regiomontanus used a circle as a key feature of his presentation of Theorem 20, even though the theorem itself is a statement about triangles and makes no mention at first about circles. This was necessary for him to connect the existing definitions of Sine and Cosine with the sides and angles of the right triangle.

In the next theorem we consider from *On Triangles* (Theorem 27), Regiomontanus solved a right triangle given only the lengths of two of its sides.

> *Theorem 27.* When two sides of a right triangle are known, all the angles can be found.
>
> If one of the given sides is opposite the right angle, that is sufficient; if not, however, we will find it also, by the preceding theorem,[10] for without it it will not be possible to handle the theorem.
>
> Thus, if triangle ABC is given with C a right angle and sides AB and AC known, then all the angles can be found.
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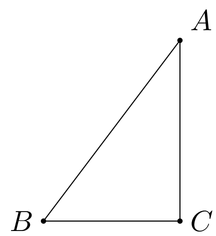
> 
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>
> When a circle is described with angle B, which the given side $[AC]$ subtends, as center and side BA as radius, then, by Thm. 20 above, AC will be the Sine of its adjacent arc, which is opposite the angle ABC that we seek . . . . Through the table of Sines, by whose neglect we can accomplish nothing in this theorem, we also ascertain the arc. Moreover, when the arc for this Sine is known, the angle opposite this arc is also given . . . . Therefore all the angles of our triangle have been found. Q.E.D.

Task 2 In the third paragraph of Theorem 27, Regiomontanus assumed that the two given known sides of the triangle are "sides AB and AC," those adjacent to the right angle. But this is not made explicit in the general statement of the theorem in the first paragraph, which refers only to "two sides of a right triangle". Explain how what he said in the second paragraph allowed him to conclude that the sides adjacent to the right angle are always determinable.

Task 3 In the final paragraph of Theorem 27, Regiomontanus noted that, given the measure of "AC, . . . the Sine of its adjacent arc, which is opposite the angle ABC that we seek," it is possible "through the table of Sines" to "ascertain the arc." Explain how this might work.

Although Regiomontanus adopted a Euclidean style of presentation in his book, he also took pains to illustrate how his theorems apply to the calculation needs of the working astronomers who were his intended audience. We next turn to the specifics of how he did this.

[10] Regiomontanus stated his Theorem 26 as follows: *If two sides of a right triangle are known, the third is directly apparent.* The justification is straightforward: he simply used the Pythagorean Theorem to determine the missing side.

4 How to solve a triangle

The following excerpt shows how Regiomontanus worked out the computation details for his Theorem 27. In this passage, Regiomontanus referred to the *whole Sine*: this is the Sine associated with a 90° angle, and is therefore the maximum Sine value possible in the user's Sine table, equal to the radius of the underlying circle. Of course, there is no right triangle with an acute angle equal to 90°, but if this acute angle is taken as large as possible, its Sine does approach the value of the radius of the underlying circle.

> The mechanics [of Theorem 27]: Take the value of the side subtending the right angle as the first number, and take the value of the side opposite the desired angle for the second number, while the value of the whole Sine is the third number. Then multiply the second by the third and divide the product by the first, for the Sine of the arc opposite the desired angle will result. From the table of Sines you may determine that arc, whose value equals the desired angle. If you subtract this [angle] from the value of a right angle, the number that remains is the second acute angle.
>
> For example, if AB is 20, AC 12, and BC 16, and the whole Sine found in our table[11] is 60000, then multiply 60000 by 12 to get 720000, which, divided by 20, leaves 36000. For this Sine the table gives an arc of approximately 36°52′. This amount is angle ABC, which subtracted from 90, finally yields about 53°8′, the size of the other acute angle.

Task 4

(a) In the first of these two "mechanics" paragraphs, Regiomontanus described a procedure for finding "the Sine of the arc opposite the desired angle" at B. Express this procedure in the form of a formula for $\text{Sin}\, B$ involving the side AB, the side AC, and the "whole Sine" $\text{Sin}\, 90°$.

(b) In the second of these paragraphs, Regiomontanus gave the dimensions of triangle ABC. Is this in fact a right triangle? How can you tell?

(c) Regiomontanus then indicated that in his table of Sines the "whole Sine" is 60000. This corresponds to choosing a circle whose radius is 60000 units in length. (This is in line with the classical setup: Ptolemy's table of chords worked with a circle whose radius contained 60 parts. Scaling up by a factor of 1000 allowed Regiomontanus higher accuracy in computation: under this scheme, more digits of the target values are kept.) Apply your formula from part (a) to verify that $\text{Sin}\, B = 36000$.

(d) Apply the procedure (or your formula from part (a)) to determine $\text{Sin}\, A$.

[11] Regiomontanus' own table of sines was not published with the first printed edition of *On Triangles*; it only arrived with the second edition in 1561.

Task 5 A scientific calculator will compute the sine, cosine, and tangent (about which we'll say more about later) of any desired angle, based on a standard circle of radius 1 unit. In this situation, the "whole sine" has the value $\sin 90^\circ = 1$.

(a) Adapt the formula from Task 4(a) to one which replaces Regiomontanus' Sines with modern sines (in a circle of radius 1).

(b) Now use the formula from part (a) to determine the values of $\sin B$ and $\sin A$ for the specific triangle from Regiomontanus' "mechanics" example.

In the final paragraph of the passage above, Regiomontanus performed a *reverse* lookup in his table of Sines: he searched for 36000 in the column of Sine values and found that it appears as the Sine of the angle $36^\circ 52'$. This, then, is what he determined the measure of angle ABC to be. We can do the same using a scientific calculator, which computes the *inverse sine* of a quantity x (denoted $\sin^{-1} x$), the *inverse cosine* ($\cos^{-1} x$), and the *inverse tangent* ($\tan^{-1} x$). Often, these calculator functions use the very same buttons as those for sine, cosine, and tangent, respectively. For instance, one can check that $\sin 30^\circ = 0.5 = \frac{1}{2}$; thus, the inverse sine of $0.5 = \frac{1}{2}$ must be 30°, that is, $\sin^{-1} 0.5 = 30^\circ$.

Task 6 Verify with a calculator that the inverse sine of the two sine values you found in Task 5 give the same angles that Regiomontanus determined at the end of the passage above.

5 Taking sides

One of several companion problems to determining the missing angles of a right triangle from knowledge of its sides—the goal of Theorem 27—is the determination of unknown lengths of sides of the triangle from knowledge of one side and one angle measure. Regiomontanus considered this problem in Theorem 29:

Theorem 29. When one of the two acute angles and one side of a right triangle are known, all the angles and sides may be found. ...

For instance, let angle ABC be given as 36° and side AB as 20 feet.[12] Subtract 36 from 90 to get 54°, the size of angle BAC. Moreover, from the table of Sines it is found that line AC is 35267 while BC is 48541 (when AB, the whole Sine, is 60000). Therefore, multiplying 35267 by 20 yields 705340, which when divided by 60000, leaves about $11\frac{45}{60}$. Thus, side AC will have 11 feet and $\frac{45}{60}$—that is, three-fourths—of one foot. Similarly multiply 48541 by 20, giving 970820, which when divided by 60000, leaves about 16 [feet] and 11 minutes,[13] the length of side BC.

[12] Regiomontanus used the foot as a unit of length here. While this is not the official US customary unit of length containing 12 inches with which many readers will be familiar, it was a common unit of measure across Europe dating to pre-Roman times. Until the introduction of the metric system in the modern era, standardization of units was elusive. Every locality used essentially its own system of weights and measures, so that the foot or the pound in one place was not necessarily the same as the foot or pound in another. Armies of clerks over the centuries were employed by governments and merchants to ensure that conversions of quantities from one system to another were correctly performed, so that taxes could be properly paid and parties were not unjustly dealt with in business transactions.

[13] In this context, one minute means one sixtieth of a foot, so 11 minutes is $\frac{11}{60}$ feet.

But if side AC is taken as 20 and everything else stays the same, AC will be 35267 and BC 48541 (when AB, the Sine of the quadrant, is 60000). Thus, to find AB, multiply 60000 by 20, producing 120000, which is divided by 35267, leaving about 34 and 2 minutes. But if it is agreed to measure side BC, multiply 48541 by 20, producing 970820, which when divided by 35267, leaves about 27 and 32 minutes. Therefore, side BC is about 27 [feet] and 32 minutes of one foot.

Task 7

(a) Draw a diagram of Regiomontanus' triangle ABC in his Theorem 29 and label the known angles as they appear in the text in the next-to-last paragraph above; he gave the hypotenuse as 20 feet. Now redraw the triangle but assign the hypotenuse, or "whole Sine," length 60000; the angles are as before, but the sides correspond to the Sines of the opposite angles. Regiomontanus stated those values in the text, so transfer them to this second diagram. Show using appropriate proportions how he found the lengths $AC = 11\frac{45}{60}$ and $BC = 16\frac{11}{60}$ for the smaller triangle (the one with $AB = 20$).

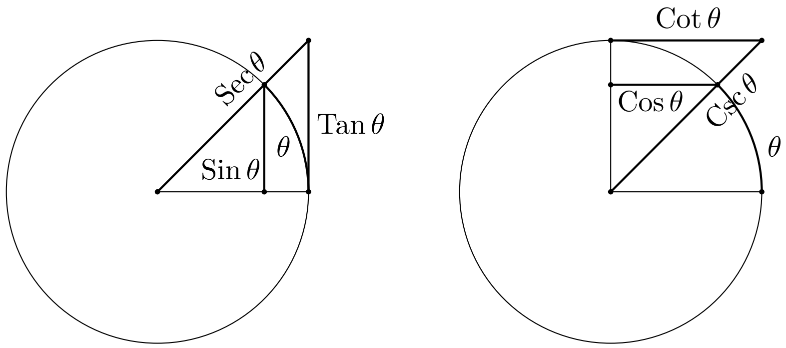
(b) Since we don't have access to Regiomontanus' table of Sines, we will use a calculator's values and modern sines to verify these values. Draw a third version of triangle ABC with the same angles, but set the length of the hypotenuse as 1 unit. Using your calculator, compute sines, rounding to five decimal places, to determine the lengths of the sides in this third diagram. Now work proportionally from this third triangle back to the first triangle to test that Regiomontanus' Sines correspond to the ones found via your calculator.

(c) Repeat the analysis of parts (a) and (b) for the example given in the last paragraph of text above. How well do Regiomontanus' Sine values compare to your calculator's sines?

6 Trigonometry Then and Now

As previously noted, Islamic mathematicians had by the eleventh century extended their trigonometric tables to include the measures of other segments related to arcs in a circle. In particular, they made use of the Tangent of the arc θ, which was the length of the line segment tangent to the circle at one end of the arc and cut off at the other end by an extended radius of the circle (see the diagram below). It is denoted more compactly as $\mathrm{Tan}\,\theta$ (or $\tan\theta$ in a circle with unit radius). The length of this extended radius was called the Secant (or secant) of the arc, written $\mathrm{Sec}\,\theta$ (or $\sec\theta$). This Secant was the hypotenuse of the right triangle whose sides were the Tangent of the arc and the radius of the circle. If we applied these same definitions to the complementary arc $90° - \theta$, we produced the Cotangent (cotangent) of the arc, denoted $\mathrm{Cot}\,\theta$ ($\cot\theta$), and Cosecant (cosecant), denoted $\mathrm{Csc}\,\theta$ ($\csc\theta$). All six of these quantities—Sine, Cosine, Tangent, Cotangent, Secant, and Cosecant—were employed to improve the computational and problem-solving power of the mathematical astronomy of the day.[14]

[14]The astronomer-mathematician al-Bīrūnī (973–1048) tabulated Tangents and Cotangents to represent the lengths of shadows in a shadow clock to help with the construction of such timekeeping devices. More on al-Bīrūnī's work can be found in the project "Al-Bīrūnī Does Trigonometry in the Shadows" from the *Convergence* article "Teaching and Learning the Trigonometric Functions through Their Origins."



One can draw a direct line from the kind of proportional calculations we have encountered in reading passages from Regiomontanus' *On Triangles* to the modern standard of defining these trigonometric quantities in terms of ratios of sides of a right triangle, a practice that is often captured by the mnemonic "SOHCAHTOA," which stands for "sine is opposite over hypotenuse; cosine is adjacent over hypotenuse; tangent is opposite over adjacent." We explore this connection in the next task.

Task 8

(a) Regiomontanus' triangle ABC, with right angle at C, is contained within the circle centered at A having radius AB, the hypotenuse of the triangle. Consequently, the whole Sine also equals the hypotenuse, and $\text{Sin}\, A$ is the length of the side opposite the angle at A. Consider a second triangle $A'B'C'$ similar to ABC but within a circle centered at A' of unit radius. If we let θ represent the measure of the equal angles at A and A' in these triangles, explain how the similarity of the triangles implies that

$$\sin\theta = \frac{BC}{AB} = \frac{\text{opposite}}{\text{hypotenuse}}.$$

(b) Argue similarly for why it must be true that

$$\cos\theta = \frac{AC}{AB} = \frac{\text{adjacent}}{\text{hypotenuse}}.$$

(c) Extend sides AB and AC to points D and E respectively, so that the line DE is parallel to BC and tangent to the circle at E. Construct the similar triangle $A'D'E'$ in the diagram with triangle $A'B'C'$. (Draw this new diagram.) Then $\text{Tan}\, A$ is DE and $\tan A'$ is $D'E'$. Using similarity of triangles, explain why

$$\tan\theta = \frac{BC}{AC} = \frac{\text{opposite}}{\text{adjacent}}.$$

After millennia of development as a tool developed by astronomers for astronomers to solve explicitly astronomical problems, we have in Regiomontanus' *On Triangles* the first treatment of trigonometry that avoided any explicit astronomical context. Instead, Regiomontanus turned to the purely geometric model of Euclidean geometry, placing the Sine, that quintessential trigonometric object, squarely within the study of plane triangles. This represented a watershed moment in the history of the subject. Indeed, in 1595, nearly a century after the first publication of *On Triangles*, the German astronomer Bartholomaeus Pitiscus (1561–1613) would publish a work with the title

Trigonometria,[15] the first appearance of the term trigonometry in print, signaling the complete separation of what used to be a collection of mathematical techniques solely for use by astronomers into a branch of mathematics with a foundation in Euclidean geometry.

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[15]Its full title was *Trigonometria: sive de solutione triangulorum tractatus brevis et perspicuus* (*Trigonometry: or, a brief and clear treatise on the solution of triangles*).

Notes to Instructors

This project is the sixth in a collection of six curricular units drawn from a Primary Source Project (PSP) titled *A Genetic Context for Understanding the Trigonometric Functions*. The full project is designed to serve students as an introduction to the study of trigonometry by providing a context for the basic ideas contained in the subject and hinting at its long history and ancient pedigree among the mathematical sciences. Each of the individual units in that PSP looks at one of the following specific aspects of the development of the mathematical science of trigonometry:

- the emergence of sexagesimal numeration in ancient Babylonian culture, developed in the service of a nascent science of astronomy;
- a modern reconstruction (as laid out in [Van Brummelen, 2009]) of a lost table of chords known to have been compiled by the Greek mathematician-astronomer Hipparchus of Rhodes (second century, BCE);
- a brief selection from Claudius Ptolemy's *Almagest* (second century, CE) [Toomer, 1998], in which the author (Ptolemy) showed how a table of chords can be used to monitor the motion of the Sun in the daytime sky for the purpose of telling the time of day;
- a few lines of Vedic verse by the Hindu scholar Varāhamihira (sixth century, CE) [Neugebauer and Pingree, 1970/1972], containing the "recipe" for a table of sines as well as some of the methods used for its construction;
- passages from *The Exhaustive Treatise on Shadows* [al-Bīrūnī, 1021; 1976], written in Arabic in the year 1021 by Abū Rayḥān Muḥammad ibn Aḥmad al-Bīrūnī, which include precursors to the modern trigonometric tangent, cotangent, secant, and cosecant;
- excerpts from Regiomontanus' *On Triangles* (1464) [Hughes, 1967], the first systematic work on trigonometry published in the West.

This collection of units is not meant to substitute for a full course in trigonometry, as many standard topics are not treated here. Rather, it is the author's intent to show students that trigonometry is a subject worthy of study by virtue of the compelling importance of the problems it was invented to address in basic astronomy in the ancient world. Each unit may be incorporated, either individually or in various combinations, into a standard course in College Algebra with Trigonometry, a stand-alone Trigonometry course, or a Precalculus course. These lessons have also been used in courses on the history of mathematics and as part of a capstone experience for pre-service secondary mathematics teachers.

In this unit, students read excerpts from a systematic work on trigonometry by Johannes Müller (1436–1476), better known today as Regiomontanus, to see how the subject began to be separated from its practical astronomical roots to become seated within Euclidean geometry as a study of triangles and the measures of their angles and sides. Examples of solving triangles are presented that ask students to familiarize themselves with calculator functions for sine and inverse sine. Connections are also made between the work of Regiomontanus and the modern practice of defining trigonometric quantities as ratios of side lengths in a right triangle.

Student Prerequisites

Students coming to this project should have been introduced to the definition of the sine, cosine and tangent of an angle, have an understanding of how similarity of triangles is manifest in proportions among corresponding sides, and know the Pythagorean Theorem. Much of this (except perhaps the trigonometric definitions) is typically within the wheelhouse of a student in their last years of high school or first year of collegiate-level mathematics.

Suggestions for Classroom Implementation

This project is designed to be completed in two class periods (of either 50 or 75 minutes in duration), along with time set aside for student preparation before each period and for homework to write up formal assignments after each period. It is recommended that the greater part of the two class days be given over to students working together in small groups to find answers to the questions posed in the tasks presented here. More detail can be found in the Sample Implementation Schedule below.

Sample Implementation Schedule (based on two 50-minute class periods)

Here is a suggested plan for implementing this project with a group of students in a course that meets in typical 50- or 75-minute periods (of course, this schedule will provide much more freedom for groups who meet in the longer period, but there is more here than can be accomplished in a single 75-minute period). The plan for each day involves a preparatory assignment meant to be completed in anticipation of the day's meeting; work to be done in the classroom with other students and under the guidance of the instructor; and follow-up homework meant for students to either write up formal responses to tasks from the project or explore ideas for which there was insufficient time in the classroom.

Day 1

- Ask students to read Sections 1 and 2 of the project before the first class day and encourage them to take note of questions they may have regarding this overview of the history of trigonometry and the context for Regiomontanus' *On Triangles*, the primary source text from which the material for the project is taken.
- In class, ask students to share any questions they may have had as a result of their reading assignment; the ensuing discussion offers an opportunity to engage the entire class in the work of the day. When appropriate, the instructor may then want to have the primary source excerpts in Section 3 read aloud to the class (slowly and with some deliberation) to allow the students to absorb the import of the mathematical claims that the author is making there. The majority of the class time, however, should be devoted to allowing students to work in small groups on providing answers to Tasks 1–3. Communication among the students will be important, especially as they work Task 1. Do what you can to have students begin Section 4 and start work on Task 4; they will need the result of part (a) of this task to begin Task 5, which can be assigned for homework.
- Have students write up solutions to any portion of the first three tasks you wish for them to reflect on (or want to use as as assessment of their work for the day).

Day 2

- The chief homework for the time between classes should be to have students concentrate on working through all of Tasks 4 and 5.
- Clear up questions students may have about their work on Tasks 4 and 5 at the beginning of the period. Work on Task 6 will focus on how to use a calculator to determine sines and inverse sines, an important skill that not all students will find familiar. They should also be able to complete most of Task 7 during the class period.
- Feel free to ask students to write up solutions to any of the tasks, or portions thereof, which they worked on during the classroom period and which you may want to use to assess their work. Section 6 and Task 8 can also be assigned as a way for students to connect their work with Regiomontanus' trigonometry in this project and modern trigonometry.

LaTeX code of this entire PSP is available from the author by request to facilitate implementation. The PSP itself may also be modified by instructors as desired to better suit the goals for their course.

Recommendations for Further Reading

Instructors who want to learn more about the history of trigonometry are recommended to consult Glen Van Brummelen's masterful *The Mathematics of the Heavens and the Earth: The early history of trigonometry* [Van Brummelen, 2009], from which much of this work took inspiration.

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