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Connecting Research to Teaching

Do Teachers Need to Incorporate the History of Mathematics in Their Teaching?

THE MERITS OF INCORPORATING HISTORY INTO mathematics education have received considerable attention and have been discussed for decades. Still, before taking as dogma that history must be incorporated in mathematics, an obvious question is, Why should the history of mathematics have a place in school mathematics? Answering this question is difficult, since the answer is subject to one's personal definition of teaching and is also bound up with one's view of mathematics. Fauvel's (1991) list of fifteen reasons for including the history of mathematics in the mathematics curriculum includes cognitive, affective, and sociocultural aspects. My purpose in this article is not to provide complete and satisfactory answers but rather, on the basis of theoretical arguments and empirical evidence, to attempt to pinpoint worthwhile considerations to help high school teachers think about what history really can do for the curriculum and for their teaching. On the basis of Fauvel's list and other scholars' arguments, I propose five reasons for using the history of mathematics in school curricula:

History increases motivation and develops a positive attitude toward learning

- History can help increase motivation and helps develop a positive attitude toward learning.
- Past obstacles in the development of mathematics can help explain what today's students find difficult.
- Historical problems can help develop students' mathematical thinking.
- History reveals the humanistic facets of mathematical knowledge.
- History gives teachers a guide for teaching.

History can help increase motivation and helps develop a positive attitude toward learning

As sometimes taught, mathematics has a reputation as a "dull drill" subject, and relevant studies report a steady decline in students' attitudes toward the subject through high school. The idea of eliciting students' interest and developing positive attitudes toward learning mathematics by using history has drawn considerable attention. Many mathematics education researchers and mathematics teachers believe that mathematics can be made more interesting by revealing mathematicians' personalities and that historical problems may awaken and maintain interest in the subject. By comparing two college algebra classes, McBride and Rollins (1977) probed the effects of including the history of mathematics and found a significant improvement in the students' attitudes toward mathematics when history was included. Philippou and Christou (1998) also reported that prospective teachers' attitudes and views of mathematics showed radical change after they took two history-based mathematics courses in a preparatory program. One teacher responded,

> History of mathematics provided me with a variety of interesting new experiences. . . . Through the journey I realize that mathematics has always been and continues to be a very useful subject. . . . The course showed me that mathematics is, at least sometimes, a human activity. I felt more confident when I realized that even great mathematicians did mistakes as I frequently do. (Philippou and Christou 1998, p. 202)

Contrary to the previously cited studies, Stander (1989) conducted two short-term experiments along this line but found that studying the history of mathematics had no significant effect on improving

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students' attitudes toward mathematics. The outcome indicates that using history only for sake of using history is superficial, as well as impractical.

Past obstacles in the development of mathematics can help explain what today's students find difficult

During the development of mathematical ideas, certain concepts were slowly recognized by mathematicians. It is reasonable to assume that today's students would also encounter difficulties when they begin to learn these concepts. For instance, the concept of function is taught to students as early as middle school, yet many students in high school (and even college students) hold incomplete and inappropriate ideas about this concept (Carlson 1998; Williams 1991).

Generally speaking, the beginnings of an implicit use of functions can be traced back to the ancient Babylonians. The earliest explicit recognition of the concept of function did not appear until the time of Nicole Oresme in the fourteenth century. James Gregory gave the first explicit, although incomplete, definition of *function* in 1667. Johann Bernoulli and Leonhard Euler systematically investigated the theory of function, yet both failed to distinguish between *function* and *value of function*. Their statements did not indicate that they recognized the uniqueness of function value. The concepts of *domain* and *range*, terms commonplace in modern textbooks, did not come into play until the late nineteenth century. We must be aware that the present definition of *function* is a result of long-term historical evolution. Students' negative outlook toward the formal definition of *function* is therefore not difficult to understand.

A moderate or convenient mathematical notation can assist our thinking in understanding mathematical concepts (Pólya 1945), whereas one of the major obstacles in learning algebra is the difficulty in using and understanding the meaning of mathematical symbols. History may also explain students' troubles in this respect (Avital 1995). As *A History of Mathematical Notations* (Cajori 1928) demonstrates, the evolution of mathematical notation was sluggish and played a significant role in developing mathematical ideas. Ancient Greek mathematics did not go beyond geometry, partially because the Greeks did not recognize the enormous contribution that using the alphabet could make to increase the effectiveness and generality of algebraic methodology (Kline 1972). The decline of ancient Chinese mathematics was also partly caused by the absence of a simple and effective symbolic system. Knowing the historical struggle to pick suitable notations can increase teachers' comprehension of students' barriers to symbolic understanding.

Many people view mathematics as a rigid, dry subject, particularly because of its rigorous and abstract features. Failure to appreciate the rigor and abstraction of mathematics may be based on an individual's mathematical maturity. Students probably do not appreciate the necessity of rigor unless they have accumulated enough appropriate experience. In this regard, a knowledge of history can give teachers and students a feeling for how standards of rigor evolved through the centuries (Arcavi 1991). What today is regarded as a nonrigorous mathematical argument was widely accepted without doubt centuries ago. Many calculus students become frustrated in grasping the formal ε-δ definition of limit. The modern concept of limit eluded several outstanding mathematicians in history; thus, expecting students to comprehend and freely use the formal ε-δ definition of limit within a short period of time is probably naive. Cornu (1991) indicates that students' cognitive obstacles may reflect the historical difficulty in the development of the concept of limit.

Historical problems can help develop students' mathematical thinking

Historical problems can help develop students' mathematical thinking

The idea of using historical mathematics problems in teaching has recently received considerable attention among scholars. In contrast to telling stories to attract students' interest and improve their attitudes, using historical problems in class has the advantage of improving students' attitudes about mathematics, as well as improving their understanding of mathematics. Many mathematical concepts have evolved and have been revised through the ages. The wisdom behind these great endeavors may provide insight into the essence of mathematical thinking. As Ernest says, "Mathematicians in history struggled to create mathematical processes and strategies which are still valuable in learning and doing mathematics" (1998, p. 25). Mathematical thinking is a combination of complicated processes: guessing, induction, deduction, specification, generalization, analogy, formal and informal reasoning, verification, and so on. Yet modern textbooks usually present mathematical concepts in a neat and polished format that "hides the struggle, hides the adventure. The whole story vanishes" (Lakatos, 1976, p. 142). By posing historical problems and analyzing the approaches by mathematicians of previous eras, students can better understand mathematical thinking and appreciate its dynamic nature. Siu (1995a, 1995b) discusses numerous examples of Euler's approaches to solving problems to explain how Euler's mind worked. For instance, in solving the problem of the seven bridges of Königsberg, Euler illustrated how generalization and specialization complement each other, introduced good notation, broke the problem into subproblems, and reassembled them to obtain a solution to the problem. These typical traits in the work of mathematicians are certainly worth pointing out to students.

In addition to presenting single typical solutions, demonstrating multiple methods for a particular problem provides another effective way to teach problem solving and develop mathematical insights (Swetz 1995). Alternative solutions for particular historic problems from different persons, time periods, and cultures can be assembled and assigned as exercises for students to contrast and compare. Students thus can be advanced from knowing to understanding, even to appreciating, these approaches.

History reveals the humanistic facets of mathematical knowledge

Before introducing the concept of integration, I prefer to ask students to propose a method for deriving the area of a circle without using the formula. I ask them to imagine themselves as middle schoolers and solve the problem merely by employing basic geometric and algebraic knowledge. They usually begin with complaining about the restriction, yet they come up with diverse approaches. After they present their solutions, I show solutions proposed by Archimedes and Liu Hui and ask students to compare the different methods of the two ancient figures. Most students are impressed by Archimedes' inconceivable idea for converting a circle into a right triangle. As shown in **figure 1,** Archimedes seemingly regarded the circle as a combination of infinitely many concentric circles and then straightened the circumferences of all concentric circles to form a right triangle by stacking them. Students are also impressed by Liu Hui's manner of partitioning a circle into infinitely many regular polygons and rearranging them to form a parallelogram, as shown in **figure 2.** Some students are surprised to find that their ideas are close to the ideas of these mathematics masters.

Calculating the sum of the harmonic series

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$$

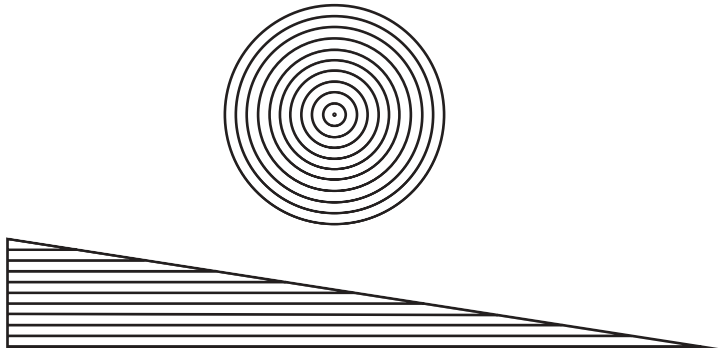

Fig. 1
Archimedes' method for deriving the area of a circle

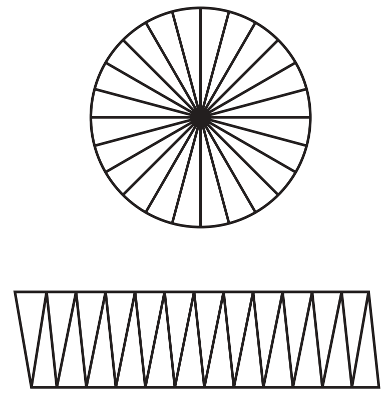

Fig. 2
Liu Hui's method for deriving the area of a circle

may also serve as a good example. The fact that the sum is infinite frequently astonishes students. Before uncovering the secret, I encourage students to explore and discover facts on their own; then I delineate the approaches employed by such mathematicians as Johann Bernoulli, Nicole Oresme, and Pietro Mengoli (as described in Dunham [1990]). After learning about three different methods, students can better appreciate the intrinsic nature of each approach. As Dunham indicates, Bernoulli's approach is trickier, Oresme's idea is clear and concise, and the beauty of Mengoli's method is its self-replicating nature. In this fashion, students learn to reject the stereotyped thinking that problems always have only one rigid and strict method of solution. As Siu (1993) indicates, multiple approaches collected from history do not merely convince students but can enlighten them.

History reveals the humanistic facets of mathematical knowledge

Considerable research suggests that many students believe that mathematics is fixed, rather than flexible, relative, and humanistic. Mathematicians' polished style in published mathematics usually eliminates the human side of grappling, of perseverance, of the ups and downs experienced on the road to final achievement (Avital 1995); and mathematics teachers pass on neatly deductive formats to students without modification. The National Council of Teachers of Mathematics proposes that helping students learn the value of mathematics is an a priori goal of school mathematics (NCTM 1989) and that all students should develop an appreciation of mathematics as being one of the greatest cultural and intellectual achievements of humankind (NCTM 2000). Yet not much has been done to achieve these objectives. By virtue of its logical and deductive traits, mathematics is typically deemed the most reliable and certain body of knowledge among all school subjects. Nevertheless, history reveals that this widely accepted impression is questionable. The history of mathematics consistently highlights the fact that the initial driving forces of mathematical knowledge are plausible conjectures and heuristic thinking; logical arguments and deductive reasoning later come into play. Acceptance or rejection of a concept is mainly tied to mathematicians' beliefs about what mathematics should be. These beliefs can be illogical, even metaphysical. Examples like the Pythagoreans' rejection of irrational numbers, Kronecker's objection to an infinite number of real numbers, and Cauchy's denial of complex numbers indicate illogical and irrational aspects of mathematical progress. Actually, in the early 1800s, no branch of mathematics was logically secure (Kline 1980). The history of mathematics notes human intellectual adventure in mathematical ideas, thus manifesting limitations of the human mind.

In addition to augmenting students' grasp of mathematical thinking, using historical problems humanizes mathematics by illustrating mathematicians' struggles in attacking problems and establishing concepts. Students are pedagogically enlightened when they realize that such problems are not created in a vacuum and more important, that mathematicians make mistakes too. These recognitions have not only cognitive merit but also affective merit. The importance of introducing humanistic aspects of such knowledge in education can be best summarized by Tymoczko's argument:

> It took human beings thousands of years to progress to the mathematical level of today's high school students, and perhaps teachers should mention this to students. . . . Educators ignore humanistic mathematics at their peril. Without it, educators may teach students to compute and to solve, just as they can teach students to read and write. But without it, educators can't teach students to love or even like, to appreciate or even understand, mathematics. (Tymoczko 1993, pp. 12–14)

History gives teachers a guide for teaching

The teacher always needs to determine the best approach of assisting students in grappling with and understanding ideas. History is one valid approach (Katz 1997). In responding to the question of whether history is important in mathematics teaching, Morris Kline indicates,

> I definitely believe that the historical sequence is an excellent guide to pedagogy. . . . Every teacher of secondary and college mathematics should know the history of mathematics. *There are many reasons, but perhaps the most important is that it is a guide to pedagogy.* [italics added] (Albers and Alexanderson 1985, p. 171)

Kline's argument explicitly and clearly delineates the chief rationale for using history in mathematics teaching.

Integrating history into school mathematics curricula not only helps improve students' attitudes and enhance higher-level thinking, but it also helps expand teachers' understanding of the nature of mathematical knowledge. Along with the growth in their understanding of "real mathematics," that is, the dialectical nature of mathematics in addition to its deductive nature, teachers are expected to restructure their beliefs about mathematics. This restructuring may in turn affect their thinking about curriculum design and instructional behavior. Planning curriculum involves far more than choosing the content to be taught. Teachers must decide the instructional sequence and the methods to use in teaching the content.

In this respect, Pólya was convinced that the "genetic principle" offers an important guide. By "genetic principle," Pólya means retracing the great steps of the mental evolution of the human race. Pólya (1965) indicates that understanding how the human race has acquired knowledge of certain facts or concepts puts us in a better position to judge how a human child should acquire such knowledge. The German mathematician Otto Toeplitz (1963) also proposed that a genetic approach is best suited to bridge the gap between high school and college mathematics:

> Follow the genetic course, which is the way man has gone in his understanding of mathematics, and you will see that humanity did ascend gradually from the simple to the complex. . . . Didactic methods can thus benefit immeasurably from the study of history. (Toeplitz 1963, p. vi)

History gives teachers a guide for teaching

For instance, the approaches that ancient mathematicians used in deriving the area of a circle can demonstrate a wide variety of mathematical thinking.

Speaking of the idea of incorporating the history of mathematics in mathematics teaching, we should not neglect a question that many teachers would ask: How can a teacher incorporate the evolution of mathematics concepts and cover all the required curriculum in the short time that we have with students? My answer is that teaching the history of mathematics is teaching mathematics itself, too. The history of mathematics is better treated as part of the lesson plans, not as an "extra" activity. After participating in a workshop, one teacher's reaction to incorporating the history of mathematics in teaching was as follows:

> When a colleague asks me if and how to use history I answer: Do not *talk* about the history of mathematics in your classroom, but do it, use it! Use historical problems in your teaching for reasons of variety and to give your pupils something extra! The extras that historical problems bring

to your pupils are historical insights and mathematical insights. (Furinghetti, 1997, p. 56)

I am convinced that, as Furinghetti indicates, a good knowledge of the history of mathematics may foster pedagogical creativity for integrating history into mathematical activities.

IS INCLUDING THE HISTORY OF MATHEMATICS IN MATHEMATICS TEACHING EFFECTIVE?

A panel discussion, “On the Role of the History of Mathematics in Mathematics Education,” at the second International Conference on the Teaching of Mathematics (ICTM-2), held on Crete in July 2002 addressed the role of the history of mathematics in education. Following the panel’s reports, an American mathematics educator raised a critical question: “Is there any evidence showing that including the history of mathematics is effective in the teaching of mathematics?” Answering this question is difficult for anyone who advocates the importance of including history in the mathematics curriculum. We have to clarify one critical conception before answering this question. Namely, what is meant by “effective in the teaching of mathematics”? If it means improving students’ performance on standardized examinations, my attitude would be reserved. To my best knowledge, no empirical study indicates that learning the history of mathematics helps students perform better on traditional tests. Although studying the history of mathematics may improve students’ attitudes toward mathematics, the linkage between attitude and achievement is neither linear nor straightforward.

Yet if effectiveness means developing students’ views of thinking and further improving their learning behavior, I am convinced that including the history of mathematics in the curriculum can help. After experiencing a problem-based course that used a historical approach, many Taiwanese students were likely to hold active views about mathematical thinking and were able to demonstrate multiple approaches to problems (Liu 2002). Particularly, when learning about “peculiar” methods used by ancient mathematicians, those students better appreciated the role of imagination in problem solving, and some students were more willing to think and try the problems. After seeing Archimedes’ derivation of the area of a circle, one of my students, who had initially emphasized the deductive nature of mathematics, reconsidered his view:

Students better appreciated the role of imagination in problem solving

> I consider imagination more important [than logical thinking] because of Archimedes. I feel he is so strange. He derived the volume of a sphere by means of a lever. . . . How did he think of it? Plus, he transformed a circle into a triangle. I feel his imagination is quite strange.

That response is typical of those of students in my class. Nevertheless, the empirical evidence accumulated thus far is insufficient for us to conclude what history can or cannot do for teachers and students. The International Study Group on the Relations between History and Pedagogy of Mathematics (HPM) is attempting to delineate a role for the history of mathematics to play in school teaching. With cooperation between researchers and teachers, we hope that a clear picture of that role can be drawn in the near future.

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