English Translation of Clairaut's
Quatre Problèmes sur de Nouvelles Courbes

Abstract

Alexis Clairaut, born in 1713 to mathematician and teacher Jean-Baptiste Clairaut and mother Catherine, was a mathematician who showed promise from a very young age. In 1726 he presented on four new families of curves and their properties to the French Royal Academy of Sciences. Clairaut published these findings in 1734 as "Quatre Problèmes sur de Nouvelles Courbes" ("Four Problems on New Curves") in the fourth volume of the journal of the Royal Prussian Academy of Sciences, *Miscellanea berolinensia ad incremental scientiarum*. Each of the four families of algebraic curves that he investigated was partly motivated by the classical Greek problem of finding mean proportionals between two given line segments. Clairaut also investigated the analytic properties of his curves by finding tangents, inflection points, and quadratures.

Clairaut's 1734 paper, written and published in French, has not yet been translated to English. We present a dual language edition—French and English—to make Clairaut's paper accessible by a modern audience. The French edition is available via the companion *Convergence* article "The Four Curves of Alexis Clairaut." Both versions include images of Clairaut's diagrams, scanned from the copy of *Miscellanea berolinensia* owned by The Rare Book & Manuscript Library, University of Illinois at Urbana Champaign. In this English version, our copies of Clairaut's diagrams show the curve in question in color (red or blue); an interactive GeoGebra rendering of each of these diagrams can also be accessed by clicking on the figure label in the caption below the diagram.

Editorial Conventions

We chose to not attempt a literal, word-for-word translation, and have instead taken some small liberties to give one with clearer meaning. For example, Clairaut employs a plethora of semicolons in his text. If we interpret them as periods, the result is a string of sentence fragments. If we interpret them as semicolons, it results in a massive run-on sentence. Accordingly, we have been flexible with sentence breaks, and have done so both to make the text readable and to remain true to what we think was Clairaut's intent.

For mathematical notation in the English, we have used modern styling. For example, we replace Clairaut's xx with x^2, and render surds as square roots, e.g. $\sqrt{x^2 + y^2}$ instead of $\sqrt{}x^2 + y^2$. Notes for the English version are of two varieties:

- *Translation.* These items comment on specific aspects of our translation.
- *Mathematical.* These items explicate mathematical details, such as intermediate steps not justified by Clairaut.

In both the English and French versions, we have taken the liberty of placing most of the mathematical notation on its own line. The source text prints them in-line, and this is often detrimental to the readability of complicated notation.

The mathematical expressions are formatted in two ways. Points are given in "math bold" style, e.g. the point $\mathbf{A}$ or the point $\mathbf{n}$. Algebraic quantities are given in standard *LaTeX* italics, e.g. x, y, a, m, n. This was necessary in order to distinguish the points $\mathbf{a}$ and $\mathbf{n}$ from the algebraic quantities a and n, as Clairaut used both throughout the paper. This is also advantageous because it distinguishes between the point $\mathbf{a}$, the algebraic quantity a, and the common French verb "a", a conjugation of "avoir".

I. Indeterminate Problem.

A right angle **ACQ** with a fixed point **A** (fig. 1) on one of its sides is given on a plane. We inquire as to the nature of the curve, whose property is such that the square of the perpendicular **MQ** on **CQ** is equal to the rectangle contained by the constant **AC** and the straight line **CM**, which always goes from the fixed point **C** to any point on the curve.

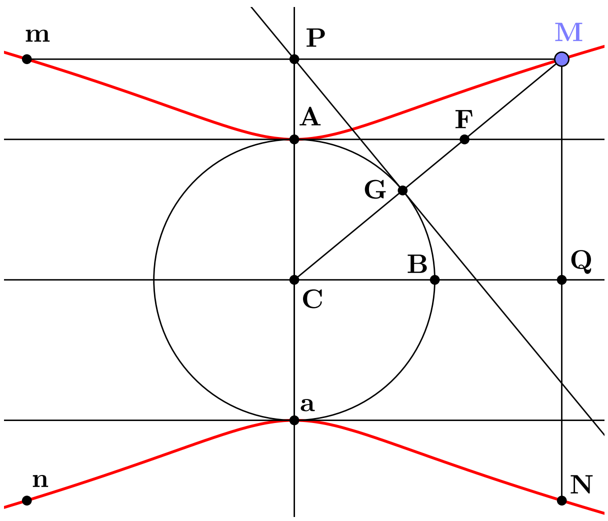
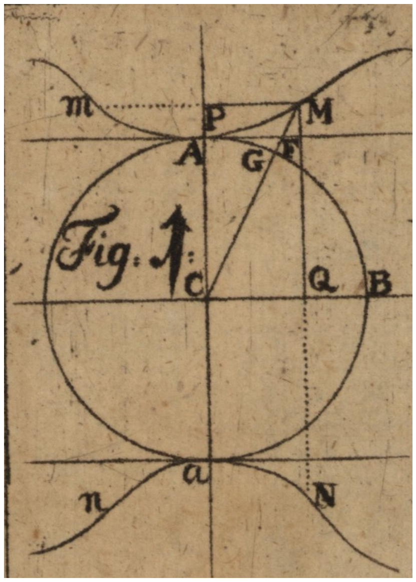


Figure 1



Clairaut's Figure 1

1. Let us assume the problem is solved; then: Let **MP** be drawn parallel to **CQ**. I will denote the length of **AC** a; the indeterminate length of **CP** or **MQ**, is denoted x; the length of **PM** or **CQ** will be denoted y; the length of **CM** will be

$$\sqrt{x^2 + y^2}. \tag{I.1}$$

And, in expressing the property of the curve, we will have

$$a\sqrt{x^2 + y^2} = x^2; \tag{I.2}$$

where in squaring each side we get

$$x^4 = a^2x^2 + a^2y^2, \tag{I.3}$$

which is an equation of the fourth degree and which makes it clear that the curve is of the third genre.

2. We find from the previous equation that

$$x = \pm\sqrt{\sqrt{a^2y^2 + \frac{1}{4}a^4} + \frac{1}{2}a^2}. \tag{I.4}$$

Hence, we see that the curve passes through both sides of the axis, having two equal and similar parts both above and below [the axis]. That is to say that there are two parts **AM** and **Am**, which form only one **mAM**. **mAM** takes its origin at **A**, which is the same distance from **C** as **a**. **a** is the origin of the other two parts, **aN** and **an**. These parts **aN**, **an** also form one, **naN**, which is equal to the first [**mAM**].

3. We clearly see that as x increases, y increases as well, and that when x is infinite, y is also infinite.

4. As the property of this curve resembles in some fashion that of a parabola, we are able to form a general equation

$$x^m = \sqrt{x^2 + y^2} \tag{I.5}$$

(taking a as unity, and $m > 1$ always) for expressing the nature to any degree whatsoever,[1] just as we make

$$x^m = y \tag{I.6}$$

to express the nature of all parabolas.

5. Because this curve also has the property that, if we describe a quarter of a circle **AGB**, which has a center **C** and radius **AC**, then **CF** is the mean proportional between **CG** and **CM**. We can therefore generalize this indefinitely by supposing that **CF** is the first of two (or, moreover, of some number n) means proportional between **CG** and **CM**. For us to have the general equation we do the following: we label **CP** x, and **PM** y; **CM** is

$$\sqrt{x^2 + y^2} \tag{I.7}$$

and **CF**

$$\frac{a}{x}\left[x^2 + y^2\right]^{\frac{1}{2}}. \tag{I.8}$$

Therefore, we have, by the property of continued proportionals,[2]

$$1 : 1 :: \frac{a}{x} : \frac{a}{x^{n+2}}[x^2 + y^2]^{\frac{n}{2}} \tag{I.9}$$

because

$$a : \sqrt{x^2 + y^2} :: a^{n+1} : \frac{a^{n+1}}{x^{n+1}}\left[x^2 + y^2\right]^{\frac{n+1}{2}}. \tag{I.10}$$

[1]It seemed redundant here to translate *à l'infini* literally.

[2]This follows from the Equation I.11. Equation I.11 follows from the properties of continued proportions; for details, refer to "Technical Notes: On the Finding of Mean Proportionals" in the companion *Convergence* article "The Four Curves of Alexis Clairaut."

This reduces to the equation

$$x^{\frac{n+1}{n}} = \sqrt{x^2 + y^2} \tag{I.11}$$

which is the sought after equation, and which becomes the same as the one we found in No. 4, if we let $\frac{n+1}{n} = m$.

6. If in the first general equation (Equation I.5) we make $m = \frac{3}{2}$ or, equivalently, we make $n = 2$ in the second [equation] (Equation I.11) (in other words, we seek the curve where **CF** is the first of the two means proportional between **CG** and **CM**), then the equation becomes

$$x^3 = ax^2 + ay^2. \tag{I.12}$$

This equation provides the following construction (fig. 2) for finding a point on the curve, supposing **CP**, x, to be determined every time. Construct a semicircle on **CP**, which will intersect **AF** at the point **F**, from which, having drawn the straight line **CF**, it will intersect **PM** at the sought after point **M**.

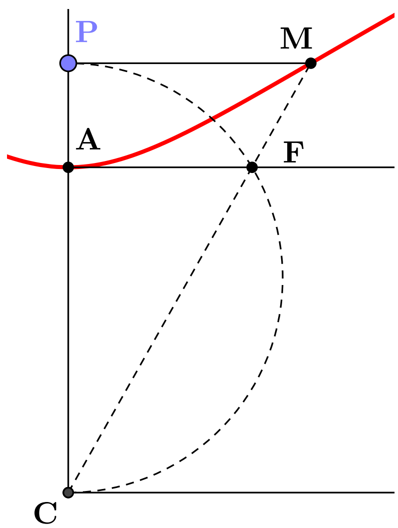
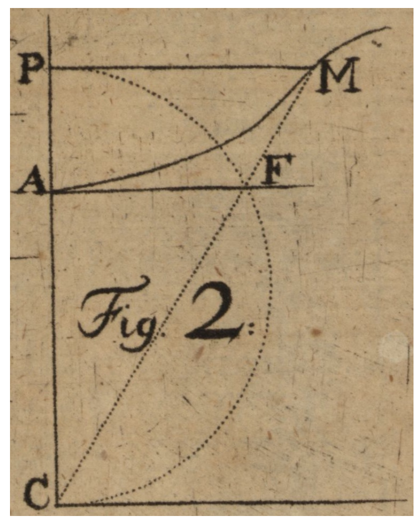


Figure 2



Clairaut's Figure 2

7. When one of these indeterminate curves having been given with its axis and its associated quarter-circle, we can, by this method,[3] find between two lines the first of the number n means proportional (provided that the number n is appropriate [appropriate given Equation I.12] for the degree of the curve given). For example, if we want to find between two given lines the first of two means proportional, we will be able to make use of the constructed curve from No. 6 in this manner.

[3]We have translated *moyen* here as "method," rather than "mean," in order to avoid confusion between "mean" as in mean proportional and "mean" as in "a way of doing something."

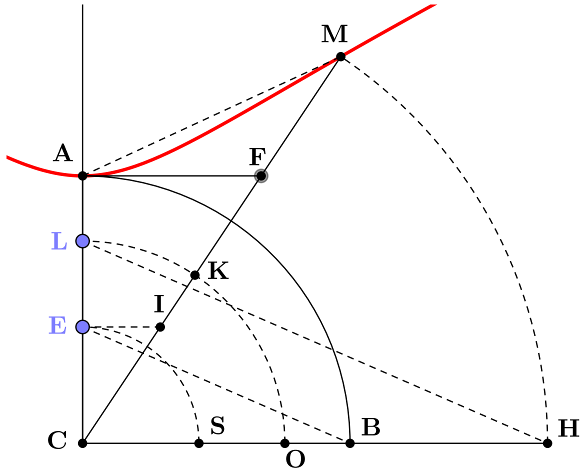
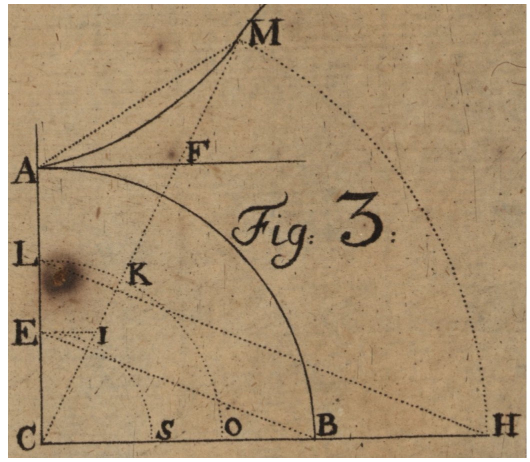


Figure 3



Clairaut's Figure 3

We will make (fig. 3) two quarter circles **ES** and **LO** on **CH**. The first **ES** has for its radius the smaller of the two given lines, and the second [**LO**], the larger. We will construct the circular arc **MH** with center **C** and radius **CH**, which will intersect the given curve **AM** at the point **M**, from which we will draw **MC**. We will also draw the lines **AM** and **EK**. It is clear by the construction that they will be parallel, and consequently that the triangles **CEK** and **CAM** are similar. Make **EI** parallel to **AF** as well, these two parallel lines **EI** and **AF** will intersect **CK**, **CM** in similar proportion. Because **CF** is the first of the two means proportional between **AC** and **CM**, **CI** will be the first of the two means proportional between the two given lines **CE** and **CL** (or **CK**).

8. In order to draw a tangent at a given point on the curve, we will use the general equation

$$x^m = \sqrt{x^2 + y^2} \tag{I.13}$$

in the following way (fig. 4).

First case: If we want to have the sub-tangent on the x-axis **AC**, by the similarity of the triangles **Mrm** and **MPT**, it will be equal to

$$y\frac{dx}{dy}, \tag{I.14}$$

and we will have, by taking the differential of the general equation,

$$x^m = \sqrt{x^2 + y^2} \tag{I.15}$$

$$mx^{m-1}\,dx = \frac{x\,dx + y\,dy}{\sqrt{x^2 + y^2}} \tag{I.16}$$

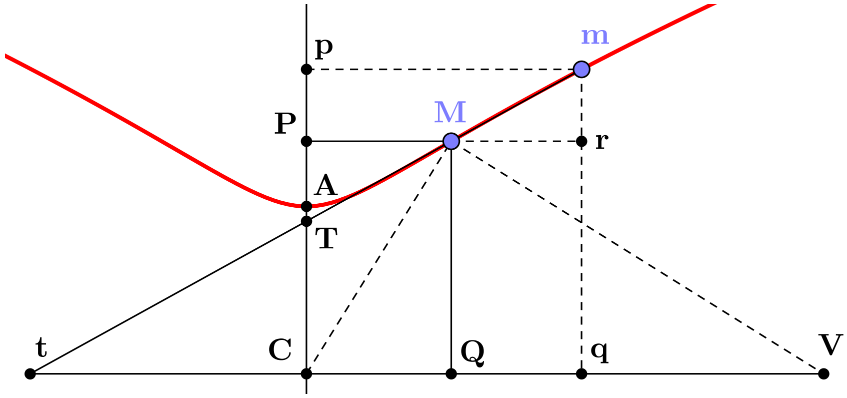
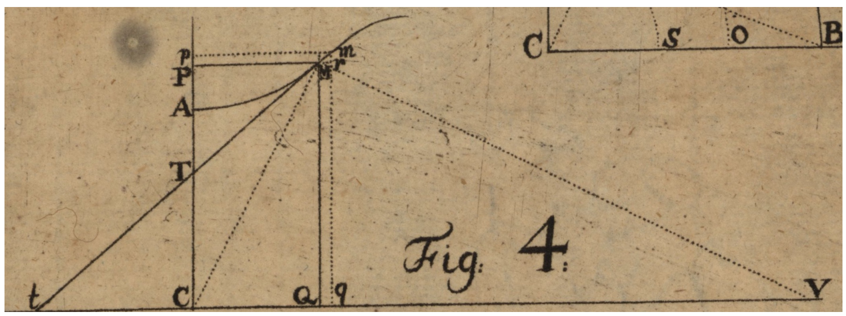


Figure 4



Clairaut's Figure 4

from which we conclude that **PT** is:

$$y\frac{dx}{dy} = \frac{y^2}{mx^{m-1}\sqrt{x^2+y^2}-x} \tag{I.17}$$

$$= \frac{y^2}{mx^{2m-1}-x} \tag{I.18}$$

$$= \frac{y^2}{\frac{n+1}{n}x^{\frac{n+2}{n}}-x}; \tag{I.19}$$

if we let m be the value $\frac{n+1}{n}$ and if $m=2$ or $n=1$ this becomes

$$\frac{a^2y^2}{2x^3-a^2x} \tag{I.20}$$

because $a=1$; if $m=\frac{3}{2}$ or $n=2$, it becomes

$$\frac{2ay^2}{3x^2-2ax}. \tag{I.21}$$

Second case: If we want to have the sub-tangent on the y-axis **CQ**, we will have by the similar triangles **mrM** and **MQt**, that this sub-tangent, which is **QT**, will be equal to

$$x\frac{dy}{dx}; \tag{I.22}$$

and, by the differential of the general equation, it will become[4]

$$\frac{mx^2+my^2-x^2}{y}=\frac{(m-1)x^2+my^2}{y}. \tag{I.23}$$

If we substitute $\frac{n+1}{n}$ for m, we get

$$\mathbf{QT}=\frac{\frac{1}{n}x^2+\frac{n+1}{n}y^2}{y}. \tag{I.24}$$

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[4]Clairaut likely proceeded by starting with Equation I.16, as in Case 1, but solving for $x\dfrac{dy}{dx}$ instead of $y\dfrac{dx}{dy}$. Doing this yields

$$\frac{x\,dy}{dx}=\frac{mx^m\sqrt{x^2+y^2}-x^2}{y}.$$

Since the general equation is $x^m=\sqrt{x^2+y^2}$, we get

$$\frac{x\,dy}{dx}=\frac{mx^{2m}-x^2}{y},$$

and since $x^{2m}=x^2+y^2$, Clairaut's equation follows.

A modern calculus student might find $x\dfrac{dy}{dx}$ in another way. The first step is to transform the general equation

$$x^m=\sqrt{x^2+y^2}$$

into

$$x^{2m}=x^2+y^2$$

and differentiate implicitly to yield

$$(mx^{2m-1}-x)\,dx=y\,dy.$$

Multiplying both sides by x and dividing both sides by $y\,dx$ yields

$$\frac{mx^{2m}-x^2}{y}.$$

Then substitute $x^{2m}=x^2+y^2$.

If we let $m = 2$ or $n = 1$, we have

$$\mathbf{QT} = \frac{x^2 + 2y^2}{y} \tag{I.25}$$

which furnishes the following construction. Erect a perpendicular **MV** on **CM** at **M**, which will intersect **CQ** at the point **V**; and we take **CV** which we will put on the other side, in order to have

$$\mathbf{QT} = \frac{x^2 + y^2}{y} + y = \frac{x^2 + 2y^2}{y}; \tag{I.26}$$

Q.E.F. If $m = \frac{3}{2}$ or $n = 2$, **QT** will be

$$\frac{x^2 + 3y^2}{2y}. \tag{I.27}$$

9. It is important to note that in the sub-tangent

$$y\frac{dx}{dy} \tag{I.28}$$

if we take away **AP** $(x - a)$ to have the part **AT**, it is positive up to a certain point, after which it will be negative. This point will be a point of inflection that we can determine in general, in the following way, by the equation from subsection 4 (Equation I.5).

$$\text{A.}\quad x^m = \sqrt{x^2 + y^2} \tag{I.29}$$

$$\text{B.}\quad y = \sqrt{x^{2m} - x^2} \tag{I.30}$$

$$\text{C.}\quad dy = \frac{mx^{2m-1}\,dx - x\,dy}{\sqrt{x^{2m} - x^2}}; \tag{I.31}$$

in which, if we take the differential a second time, we can suppose dx is constant, and replace it with the letter b.

$$\text{D.}\quad \frac{\begin{array}{c}[2bm^2x^{4m-2}\,dx - bmx^{4m-2}\,dx - x^{2m}b\,dx - 2bm^2x^{2m}\,dx \\ + bmx^{2m}\,dx + bx^2\,dx - bm^2x^{4m-2}\,dx + 2bmx^{2m}\,dx - bx^2\,dx]\end{array}}{[x^{2m} - x^2]^{\frac{3}{2}}} = ddy \tag{I.32}$$

which being equal to zero, and divided by $b\,dx$, which is to say by dx^2, will give

$$m^2x^{4m-2} + 3mx^{2m} = mx^{4m-2} + x^{2m} + 2m^2x^{2m}. \tag{I.33}$$

Dividing by x^{2m}, this is reduced to

$$m^2x^{2m-2} + 3m = mx^{2m-2} + 1 + 2m^2; \tag{I.34}$$

from which we conclude

$$x^{2m-2} = \frac{1 + 2m^2 - 3m}{m^2 - m}, \tag{I.35}$$

or

$$x^{2m-2} = \frac{2m - 1}{m}, \tag{I.36}$$

which is a general formula for determining the points of inflection of these curves, of whatever degree they are.

I have sought the quadrature of the first of these curves, which is of the third genre, in this manner; y being equal to

$$\frac{x}{a}\sqrt{x^2 - a^2}, \tag{I.37}$$

the element $y\,dx$ will be

$$\frac{x\,dx}{a}\sqrt{x^2 - a^2}, \tag{I.38}$$

and its integral **PAM** equal to

$$\frac{1}{3a}\left[x^2 - a^2\right]^{\frac{3}{2}}. \tag{I.39}$$

II. Indeterminate Problem.

The same things (fig. 5) as in the preceding section being given, we inquire as to the nature of the curve whose property is such that

$$\mathbf{CM} \times \mathbf{QM} = [\mathbf{AC}]^2. \tag{II.1}$$

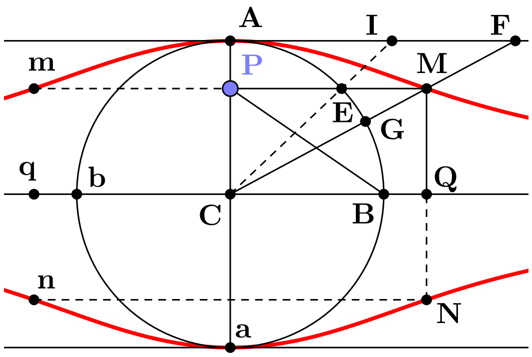
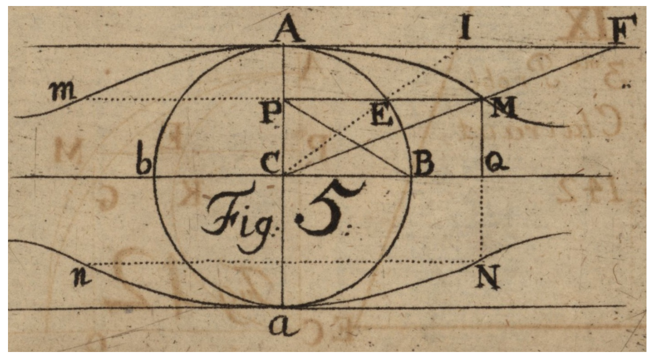


Figure 5



Clairaut's Figure 5

1. Still denoting **AC**, a; **CP** or **QM**, x; **PM** or **CQ**, y; we will also have

$$\mathbf{CM} = \sqrt{x^2 + y^2} \tag{II.2}$$

and, by the property of this curve,

$$x^4 + x^2y^2 = a^4, \tag{II.3}$$

which is an equation of the fourth degree, and which means that the curve is of the third genre.

2. It is clear from this equation:

1. That the curve passes through both sides of the axis, and that the part **Am** is similar and equal to the part **AM**.
2. That the curve has two parts equal and similar **an**, **aN** below, equidistant from **C** as the other two.
3. That as x increases, y decreases; from which we see that the two first parts are but one, which takes its origin at **A**, as well as that the second two parts are but one, which takes its origin at **a**, being as distant from **C** as **A** is. Also, that the line **qCQ** is the asymptote to the four parts of the curve **AM**, **Am**, **Na**, and **an**.

3. We further conclude from this equation the following property of the curve, that if we make a quarter circle on **AC**, then

$$\mathbf{PE} \times \mathbf{PB} = \mathbf{CP} \times \mathbf{PM}. \tag{II.4}$$

We further conclude from this that

$$x : a :: a : \sqrt{x^2 + y^2}, \tag{II.5}$$

which may give the curve a very simple construction.

4. Because the property of this curve resembles that of a hyperbola with respect to its asymptotes, we may assume a general equation

$$x^{-m} = \sqrt{x^2 + y^2} \tag{II.6}$$

to express this nature in every degree, in the same way that

$$x^{-m} = y \tag{II.7}$$

expresses the nature of every hyperbola at its asymptotes.

5. If we make a quarter circle on **CA**, it is clear that **CM** is a mean proportional between **CG** and **CF**, which shows us that we may also generalize this curve to infinity, by the means of this property, assuming that **CM** is the first of n means proportional between **CG** and **CF**. To have the general equation, we make the proportion

$$\overbrace{a}^{\mathbf{CG}} : \overbrace{\frac{a}{x}[xx + yy]^{\frac{1}{2}}}^{\mathbf{CF}} :: \overbrace{a^{n+1}}^{\mathbf{CG}^{n+1}} . \overbrace{(xx + yy)^{\frac{n+1}{2}}}^{\mathbf{CM}^{n+1}} \tag{II.8}$$

which we simplify to

$$1 : \frac{1}{x} :: a^{n+1} : \sqrt{x^2 + y^2} \tag{II.9}$$

which gives the desired equation

$$x^{-\frac{1}{n}} = \sqrt{x^2 + y^2}. \tag{II.10}$$

We can rewrite this so it resembles the equation found above (equation II.6); to do this, make $\frac{1}{n} = m$ or, equivalently, $-\frac{1}{n} = -m$.

6. If in this equation (fig. 6) we let $n = 2$, that is, if in the first equation we let $m = \frac{1}{2}$; the equation becomes

$$a^3 = x^3 + xy^2, \tag{II.11}$$

which provides the following construction for finding points on the curve, assuming x is given in each case that we want to find a point.

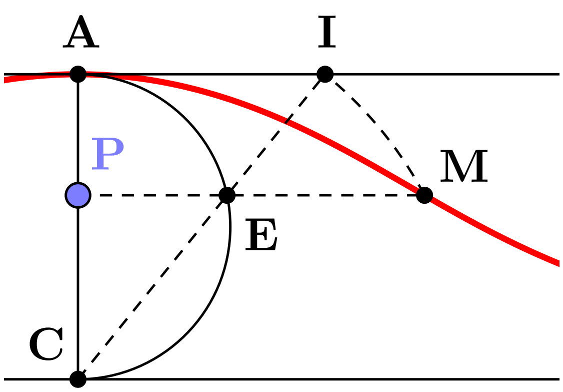
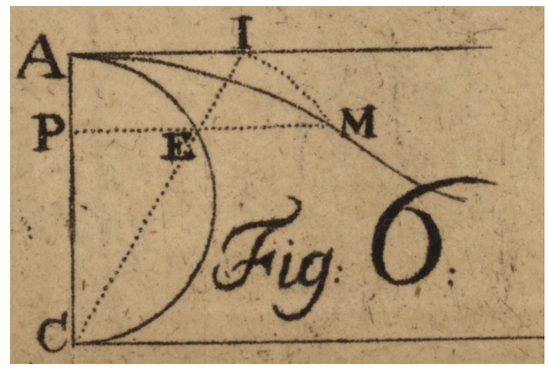


Figure 6



Clairaut's Figure 6

Make a semicircle on **AC** which intersects **PM** at **E**, from which you will draw **CEI**, and make **CM** = **CI**, which will determine the point **M** on the curve.

7. We can also, by using one of these curves, find between the two given lines the first number of a number n of mean proportionals, as one can see by mere inspection of fig. 7.

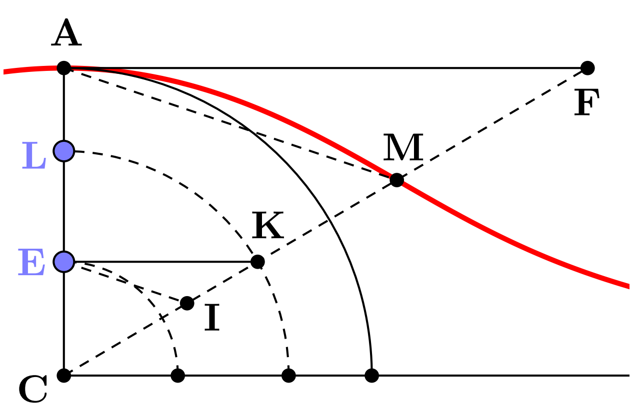
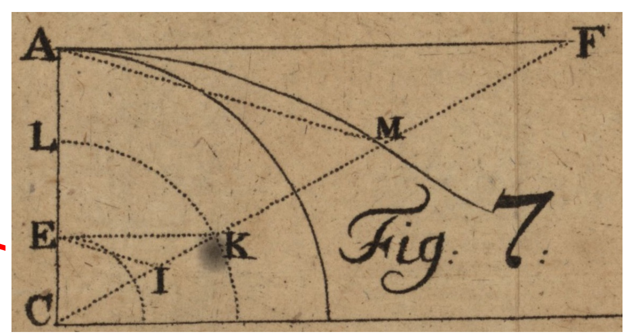


Figure 7



Clairaut's Figure 7

8. If we wish to draw a tangent from a given point on one of these curves (fig. 8), we use the general equation

$$x^{-m} = \sqrt{x^2 + y^2}. \tag{II.12}$$

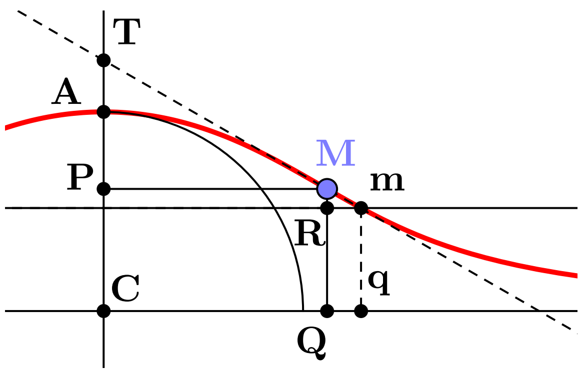
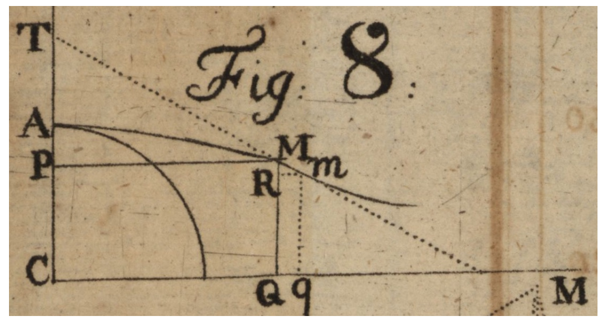


Figure 8



Clairaut's Figure 8

However, this equation is the same as that equation which we had formed in Problem I.4, except that m is negative. Therefore, it must also be that the sub-tangents are the same, with the difference m will be negative instead of positive. We will have for these sub-tangents:

$$\frac{y\,dx}{dy} = \frac{y^2}{-mx^{-m-1}\sqrt{x^2+y^2} - x} \tag{II.13}$$

and

$$\frac{x\,dy}{dx} = \frac{-mx^{-m}\sqrt{x^2+y^2} - x^2}{y} = \frac{-mx^{-2m} - x^2}{y} \tag{II.14}$$

If we substitute the value $\frac{1}{n}$ for m, we also have

$$\frac{y^2}{-\frac{1}{n}x^{\frac{n-1}{n}}\sqrt{x^2+y^2} - x} = y\tfrac{dx}{dy} \tag{II.15}$$

and

$$\frac{x\,dy}{dx} = \frac{-\frac{1}{n}y^2 - \frac{1}{n}x^2 - x^2}{y}. \tag{II.16}$$

If in these sub-tangents we make $m = 1$ or $n = 1$, we have

$$y\frac{dx}{dy} = \frac{-y^2 x}{2x^2 + y^2} \tag{II.17}$$

and

$$x\frac{dy}{dx} = \frac{-2x^2 - y^2}{y}. \tag{II.18}$$

If $m = \frac{1}{2}$ or $n = 2$,

$$y\frac{dx}{dy} = \frac{-y^2x}{\frac{3}{2}x^2 + y^2} \tag{II.19}$$

and

$$x\frac{dy}{dx} = \frac{-\frac{3}{2}x^2 - \frac{1}{2}y^2}{y} \tag{II.20}$$

which are the sub-tangents for these curves of the third and fourth degrees.

9. To determine the points of inflection of these curves, whatever degree they may be, we will use the formula

$$x^{2m-2} = \frac{2m-1}{m}, \tag{II.21}$$

(Problem I.4, Equation I.36). We make m negative in this, which will give[5]

$$x^{2m+2} = \frac{m}{2m+1} \tag{II.22}$$

which is the general formula to determine the points of inflection in these curves. For example, the curve is of the fourth degree where $m = 1$, or just as well $n = 1$, because the formula could also be

$$x^{\frac{2n+2}{n}} = \frac{1}{2+n}. \tag{II.23}$$

We then have

$$x = a\sqrt[4]{\tfrac{1}{3}}. \tag{II.24}$$

If we make $m = \frac{1}{2}$, or just as well $n = 2$, we have

$$x = a\sqrt[3]{\frac{1}{4}} \tag{II.25}$$

because $a = 1$ by assumption. Since the asymptotes of this curve resemble those of a hyperbola, and since the curve in the previous problem is quite similar to a parabola, I have given them the names *Curves of Parabolic* and *Hyperbolic Medians*.

[5]Clairaut was recycling the result from subsection I.9 here. While m was supposed positive in general for Article I, the derivation of the inflection points in subsection I.9 did not assume m had a particular sign. Hence Clairaut could take that result (which he restated here as Equation II.21) and substitute $-m$ for m. Upon simplifying, we get this equation. The reader may wish to check the computation by computing $d(dy)$ with the assumptions in Article II instead.

III. Indeterminate Problem

Let us assume (fig. 9)[6] the same things as in Article I. After having drawn **QGM** parallel to **CP** through the point **G**, we inquire as to the nature of the curve **AM**, which passes through all of the points **M** of intersection of **PM** and **QGM**. That is, the curve's property is

$$\mathbf{QG} : \overbrace{\mathbf{QO}}^{=\mathbf{CA}} :: \mathbf{QO} : \overbrace{\mathbf{QM}}^{=\mathbf{CP}}. \tag{III.1}$$

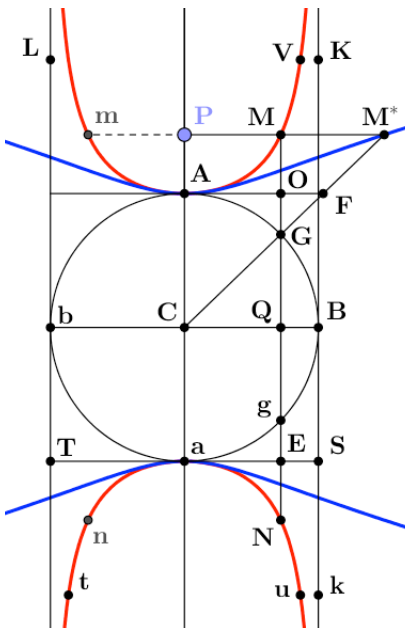


Figure 9

Clairaut's Figure 9

1. Still denoting **AC**, a; **CP**, x; **PM**, y; then **GQ** will be

$$\sqrt{a^2 - y^2}, \tag{III.2}$$

and we will have by the conditions of the problem

$$x : a :: a : \sqrt{a^2 - y^2}. \tag{III.3}$$

[6] Clairaut's figure has several errors. First, the point **m** should be on the opposite side of **P** as **M**, similar to how **n** and **N** are situated below. The curve in question in this section is the pair of curves **mAM** and **naN**. Accordingly, what Clairaut labelled as **m** here is really on the curve from Article II. We have changed our version of the figure to call this point **M***.

From this we conclude that

$$a^4 = a^2x^2 - x^2y^2, \tag{III.4}$$

which is an equation of the fourth degree, and this shows that the curve is of the third genre.

2. We conclude from this equation:

1. That the curve passes through both sides of the axis, and that the part **AM** is equal and similar to the part **Am**.
2. That the curve also has two other parts **an** and **aN**, equal and similar, and that the two first parts **Am**, **AM**, have their origin at **A**. Similarly, that the two parts **an**, **aN** begin at **a**, having the same distance from **C** as **A**.
3. That as x increases, y increases as well; and as x approaches infinity, y approaches a; by consequence, the lines **BK**, **bL**, parallel to one another and to **AC**, are asymptotes to the four parts of the curve **AM**, **Am**, **aN**, **an**.

3. We also deduce from this equation the following property of the curve, which is that[7]

$$\mathbf{CA} \times \mathbf{AF} = \mathbf{CP} \times \mathbf{PM}. \tag{III.5}$$

[7]This claim needs a bit more justification. We have

$$x : a :: a : \sqrt{a^2 - y^2},$$

from the defining property of the curve. This yields

$$x = \frac{a^2}{\sqrt{a^2 - y^2}},$$

which is an intermediate step to Equation III.4. We also note that the triangles **CAF** and **CQG** are similar, hence

$$\mathbf{AF} : \mathbf{CA} :: \mathbf{CQ} : \mathbf{GQ}.$$

Hence

$$\mathbf{AF} = \frac{ay}{\sqrt{a^2 - y^2}}.$$

Then

$$\mathbf{CA} \times \mathbf{AF} = a\frac{ay}{\sqrt{a^2 - y^2}} = xy = \mathbf{CP} \times \mathbf{PM}.$$

4. We can also generalize this curve to infinity by supposing that **OQ** is the first of n mean proportionals between **GQ** and **QM**. To derive the general equation, we will make this proportion:

$$\overbrace{(aa-yy)^{\frac{1}{2}}}^{\mathbf{GQ}} \,.\, \overbrace{x}^{\mathbf{QM}} :: \overbrace{(aa-yy)^{\frac{n+1}{2}}}^{\mathbf{QG}^{n+1}} \,.\, \overbrace{a^{n+1}}^{\mathbf{QO}^{n+1}} \,. \tag{III.6}$$

From this we conclude

$$a^{n+1} = x[a^2-y^2]^{\frac{n}{2}}, \tag{III.7}$$

which is the desired equation. If we let $n=2$, we have

$$a^3 = a^2x - y^2x, \tag{III.8}$$

which furnishes the following construction to find a point on the curve (fig. 10). Suppose x is determined in each case; draw a semicircle **CFP** on **CP**, which intersects **AF** at the point **F**. One must draw **CF**, and at the point **G** where this line intersects the quarter circle **AGB**, draw **QGM** parallel to **CP**, and then this line **QGM** will intersect **PM** at the sought after point **M**.

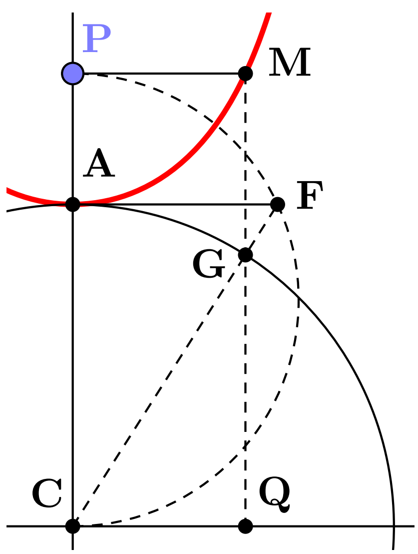
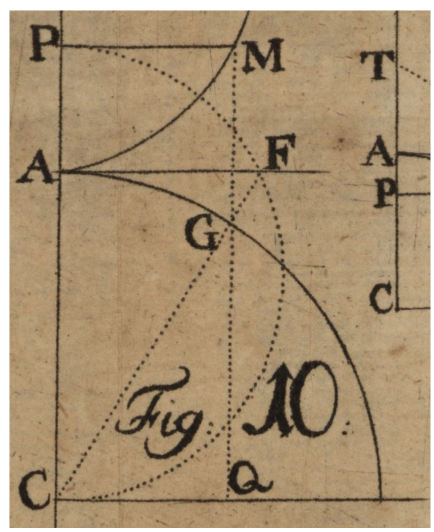


Figure 10



Clairaut's Figure 10

5. In general, in order to draw a tangent to one of these curves at a given point on it, take the differential of the equation

$$a^{n+1} = x[a^2-y^2]^{\frac{n}{2}}, \tag{III.9}$$

yielding

$$0 = [a^2 - y^2]^{\frac{n}{2}}\, dx + x\tfrac{n}{2}[a^2 - y^2]^{\frac{n}{2}-1} \times [-2y\, dy] = [a^2 - y^2]^{\frac{n}{2}}\, dx - nxy[a^2 - y^2]^{\frac{n}{2}-1}\, dy. \quad \text{(III.10)}$$

From this we conclude

$$yx[a^2 - y^2]^{\frac{n-2}{2}} n\, dy = [a^2 - y^2]^{\frac{n}{2}}\, dx, \quad \text{(III.11)}$$

which gives

$$x\frac{dy}{dx} = \frac{a^2 - y^2}{ny}. \quad \text{(III.12)}$$

This equation gives a very easy method for drawing a tangent[8] to any of these curves, because we need only take the $\frac{1}{n}$-th of the sub-tangent of the circle **AGB** at the point **G** where the perpendicular **QM** intersects it, and place it on the opposite side as the sub-tangent of the curve. For example, for a curve of the fourth degree (when $n = 1$), we have the sub-tangent equal the sub-tangent of the circle; for a curve of the third degree (when $n = 2$), we have at the sub-tangent equal to the half of that of the circle, etc. I found that the integral of the element $x\, dy$ of the first of these curves, depends on that of the circle, because

$$x = \frac{a^2}{\sqrt{a^2 - y^2}} \quad \text{(III.13)}$$

and

$$x\, dy = \frac{a^2\, dy}{\sqrt{a^2 - y^2}}. \quad \text{(III.14)}$$

[8]This passage was rather difficult to understand. What Clairaut was saying is that the sub-tangent to the curve is related to the sub-tangent to the circle. If we consider the segment **QG** in figure 10, and denote it z, then its equation is $y^2 + z^2 = a^2$. Hence the sub-tangent for the circle is

$$z\frac{dy}{dz} = -\frac{z^2}{y}.$$

The sub-tangent for the curve is

$$x\frac{dy}{dx} = \frac{a^2 - y^2}{ny} = \frac{z^2}{ny}.$$

When $n = 1$, the sub-tangents for the circle and the curve are negatives of one another; for $n = 2$ or higher the sub-tangent for the curve is obtained by dividing the sub-tangent for the circle by n, and negating the result.

From this we see that **QMAC**, which is $\int x\,dy$,[9] is equal to the product of **AC** by **AG**, which is the same as saying two times the segment **ACG**. If **CQ** becomes **CB**; **CAG** will become a quarter of the circle **ACBG**, which will make again equal to the half of the indefinite area **ACBKV** and by consequence to the region **AGBKV**.[10]

[9]This integral can be computed by letting $y = a\sin\theta$, which is obvious from the figure if we let the angle $\mathbf{ACG} = \theta_0$, measured clockwise. Hence $\mathbf{CQ} = a\sin\theta_0$, and then the integral

$$\int_0^{a\sin\theta_0} x\,dy = \int_0^{\theta_0} a^2\,d\theta,$$

which is twice the area of the sector **ACG**. Another possibility is to see that the integral in question is related to the arc length integral of the circle, since if $z = \sqrt{a^2 - y^2}$ defines a circle with radius a, then the arc length element $ds = \sqrt{1 + \left(\frac{dz}{dy}\right)^2}\,dy = \frac{a\,dy}{\sqrt{a^2 - y^2}} = \frac{x\,dy}{a}$.

[10]This region is contained by three lines (two of which are unbounded): the arc of the circle, the curve, and the asymptote.

IV. Indeterminate Problem.

Having assumed the same things (fig. 11) as in Article II after having drawn **GMQ** through **Q** parallel to **AC**, we inquire as to the nature of the curve which passes through all of the points **M**, the intersections of **PM** and of **QM**, that is, whose property is that **QG**.**QM** :: **QM**. $\overbrace{\mathbf{QO}}^{=\mathbf{AC}}$.

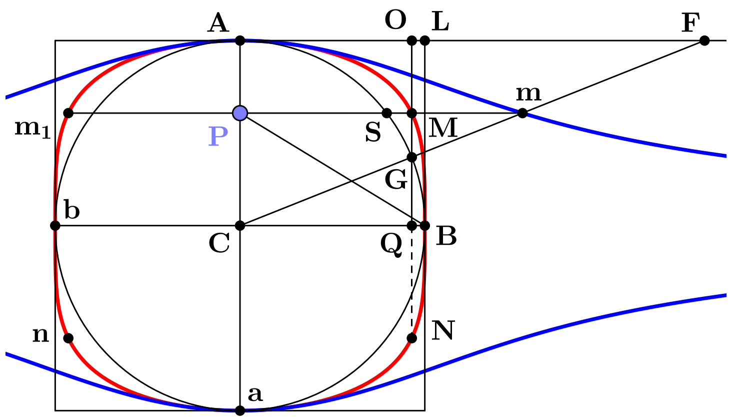
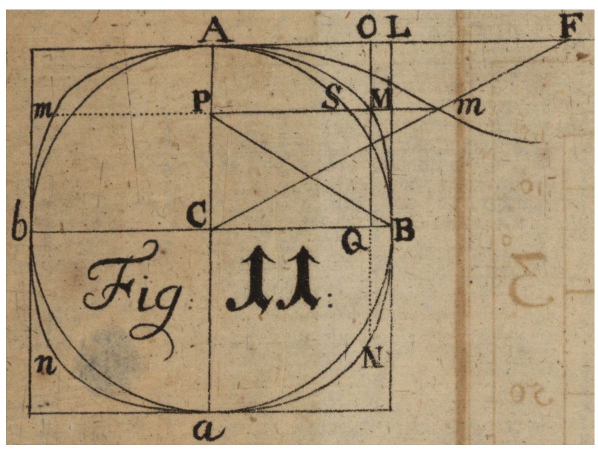


Figure 11



Clairaut's Figure 11
(The markings towards the right of the figure
are from the obverse side of the page.)

1. We denote **AC** or **QO**, a; **CP** or **QM**, x; **PM** or **CQ**, y; we have

$$\mathbf{GQ} = \sqrt{a^2 - y^2}, \tag{IV.1}$$

and by the characteristics of the problem

$$\sqrt{a^2 - y^2} : x :: x : a \tag{IV.2}$$

from which we find

$$a^4 - a^2y^2 = x^4 \tag{IV.3}$$

which is the equation of the curve. The equation being of the fourth degree shows that the curve is of the third genre.

2. From the equation we just found, we conclude:

1. That the curve passes through both sides of the axis **ACa**, and that the part **Am** is equal and similar to the part **AM**, and that the origin of the two parts is **A**.
2. That the curve passes through both sides of the axis **BCb** and that the part **an** is equal and similar to the part **aN**, and that the origin of these two parts is **a**.
3. That as x increases, y decreases,[11] from which we see that the curve is closed, and that the four parts **AB**, **Ab**, **aB**, **ab** connect the points **A**, **a**, **b**, **B**.

3. We also find this property of the curve, that:

$$\mathbf{PS} \times \mathbf{PB} = \mathbf{CA} \times \mathbf{QC}. \tag{IV.4}$$

4. We can also generalize this curve by supposing that **QM** is the first of some number n mean proportionals between **QG** and **QO**, which gives the proportion

$$\sqrt{a^2 - y^2} : a :: [a^2 - y^2]^{\frac{n+1}{2}} : x^{\frac{n+1}{2}}. \tag{IV.5}$$

The proportion reduces to

$$1 : a :: [a^2 - y^2]^{\frac{n}{2}} : x^{n+1} \tag{IV.6}$$

from which we conclude

$$x^{n+1} = a[a^2 - y^2]^{\frac{n}{2}} \tag{IV.7}$$

which is the general equation.

5. If $n = 2$ the equation becomes

$$x^3 = a^3 - ay^2 \tag{IV.8}$$

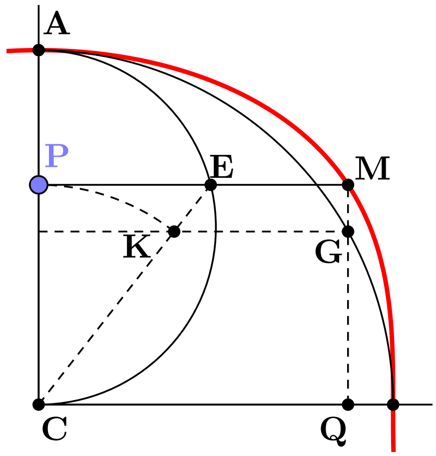
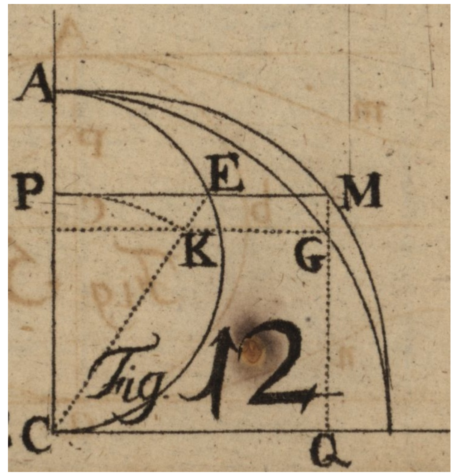


Figure 12



Clairaut's Figure 12

in which **QM** represents the first of the two mean proportionals between **QG** and **QO**. To construct this, make a semicircle **AEC** on **AC** (fig. 12), which will intersect **PM** at the point **E**, which is the endpoint of **CE**. Next make **CK** = **CP**. Through the point **K**, draw **KG** parallel to **PM**, which will intersect the quarter circle at point **G**. Having drawn **GM** perpendicular to **PM**, the point **M** on the curve will thereby be determined.

Thus, each particular equation can be used to construct the curve which corresponds to it; and having once constructed one of these curves, it can be used to find the first of as many mean proportionals n between the two given lines, where the exponent n corresponds to the genre of the curve. For example, in using the curve which we just constructed, where we had assumed $n = 2$, we can thereby find between the two given lines **p**, **q** the first of the two mean proportionals. Thus let the curve be given, that has been constructed with its quarter circle **AMB** (fig. 13):[12]

1. Extend the radii **CA**, **CB**, and describe the quarter circle **DH** having as its radius the larger of **p** and **q**.

2. Make the squares **ATB**, and **CDKH**.

3. Draw the line **Ik** through the point **I**, parallel to **CH**. And from the point **k**, where the line intersects the quarter circle, draw **Lk** parallel to **CD**.

4. Make **HI** equal to the smaller of the two given lines and draw from point **I**, the line **Ik** parallel to **CH** and through the point **K**. Where the line intersects the quarter circle, draw **Lk** parallel to **CD**.

[11]Note that the generalizations of this curve, starting in section 4, do not necessarily have this property.

[12]Although Clairaut's published paper did not explicitly reference figure 13 in this subsection, we have added the missing reference for the sake of clarity. The figure itself is included in his paper.

5. From the point **L** and **K**, draw lines to the center **C**. The line **AT** will be cut at the point **O** in the same ratio as **DK** is by the point **L**.

6. Draw **QO** parallel to **CA** which will be intersected by the curve at point **M**.

7. Draw the line **CMV** through the points **C** and **M**, which will intersect **LR** at point **V**, so that **RV** will be the first of the two mean proportionals between **kR** or **IH** and **LR** or **KH**, similarly, **QM**, by the property of the curve, is the first of the two mean proportionals between **QG** and **QO**.

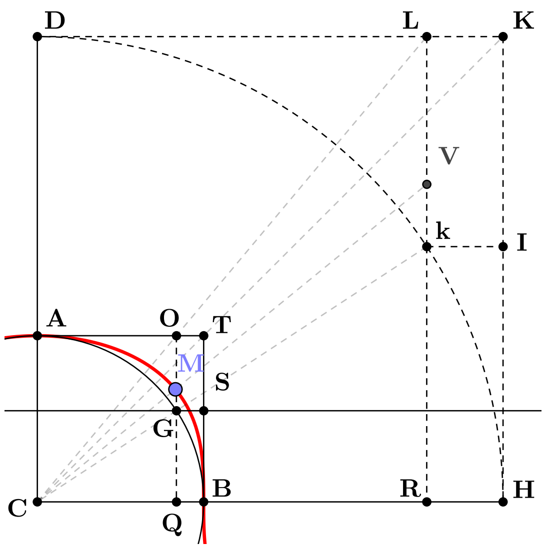
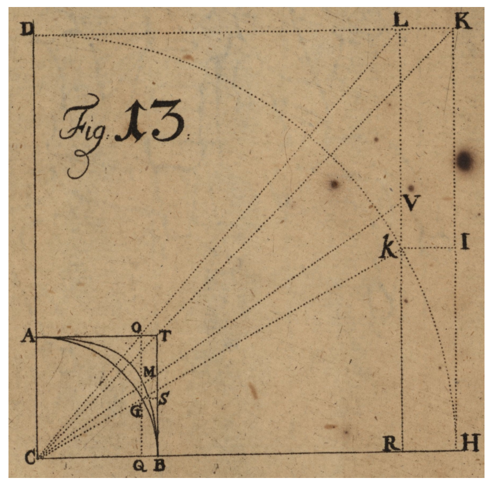


Figure 13



Clairaut's Figure 13

6. To draw a tangent to these curves in general, we will take differentials of the equation

$$x^{n+1} = a[a^2 - y^2]^{\frac{n}{2}} \tag{IV.9}$$

which will be

$$[n+1]x^n\,dx = a\frac{n}{2}[a^2 - y^2]^{\frac{n}{2}-1} \times [-2y\,dy]. \tag{IV.10}$$

Reducing and multiplying by x, and dividing[13] by dx,

$$\frac{[n+1]x^{n+1}}{ay} = -n(a^2 - y^2)^{\frac{n}{2}-1}x\frac{dy}{dx} \tag{IV.11}$$

[13]Note that Clairaut actually divided by $ay\,dx$, but said only that he divided by dx.

from which we can conclude

$$-\frac{n+1}{n}\frac{x^{n+1}}{a[a^2-y^2]^{\frac{n}{2}}}\frac{a^2-y^2}{y}=x\frac{dy}{dx} \tag{IV.12}$$

which can be reduced to

$$-\frac{n+1}{n}\times\frac{a^2-y^2}{y}=x\tfrac{dy}{dx} \tag{IV.13}$$

which is sub-tangent to the curve on the axis **CQ**. Since

$$\frac{a^2-y^2}{y} \tag{IV.14}$$

is the sub-tangent of **AGB** at point **G**, it follows, that the sub-tangent of the curve is to that of the circle as $n+1$ to n, which gives an easier means of drawing a tangent. If we want to know the ratio of the sub-tangent of the first of these curves of the third genre with that of this circle, we see that it is double, because upon letting $n=1$ we have

$$x\frac{dy}{dx}=\frac{2a^2-2y^2}{y}. \tag{IV.15}$$

7. We note (fig. 14)[14] that all of the lines like **CM**, which I will call radii of the curve, are increasing from the point **A**, until a certain point **M**, and that after these, the radii diminish, so that at the point **M**, the radius **CM** is the largest.

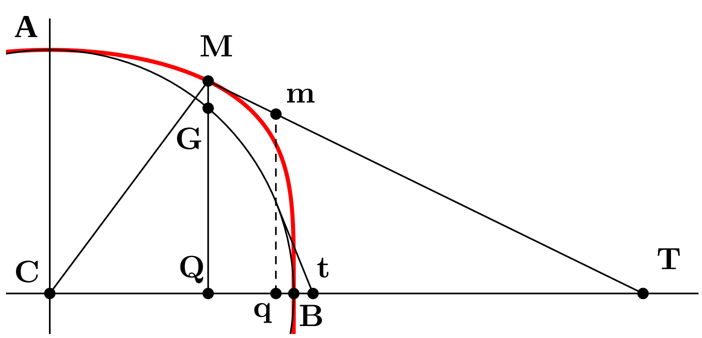
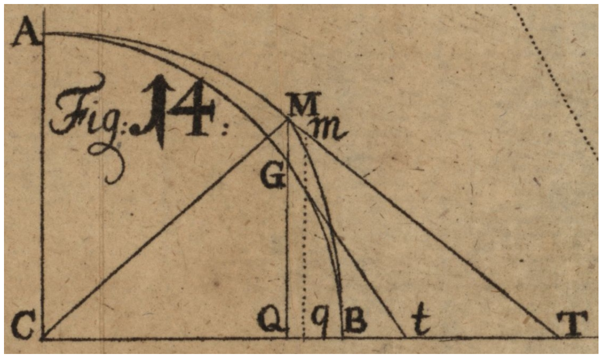


Figure 14

To find this point **M**, it is necessary to express **CM** symbolically as follows: **QM** is y, and **QC** is x, where

$$x=a^{\frac{1}{n+1}}\times[a^2-y^2]^{\frac{n}{2n+2}}, \tag{IV.16}$$

[14]Although Clairaut's published paper did not explicitly reference figure 14 in this subsection, we have added the missing reference for the sake of clarity. The figure itself is included in his paper.



Clairaut's Figure 14

and

$$\mathbf{CM} = \sqrt{x^2 + y^2}, \tag{IV.17}$$

which will be equal to

$$\sqrt{a^{\frac{2}{n+1}} \times [a^2 - y^2]^{\frac{n}{n+1}} + y^2}. \tag{IV.18}$$

Therefore, the differential is

$$\frac{a^{\frac{2}{n+1}}[a^2 - y^2]^{\frac{n}{n+1}-1}\left[\frac{n}{n+1}(-2y\,dy)\right] + 2y\,dy}{2\sqrt{a^{\frac{2}{n+1}}(a^2 - y^2)^{\frac{n}{n+1}} + y^2}} \tag{IV.19}$$

which, being set equal to zero, will give

$$a^{\frac{2}{n+1}}\frac{n}{n+1}[a^2 - y^2]^{\frac{-1}{n+1}}[-2y\,dy] + 2y\,dy = 0. \tag{IV.20}$$

This we can reduce to

$$a^{\frac{2}{n+1}}\frac{n}{n+1} = [a^2 - y^2]^{\frac{1}{n+1}}, \tag{IV.21}$$

and when we raise both sides to the power of $n + 1$, we have

$$a^2\frac{n^{n+1}}{[n+1]^{n+1}} = a^2 - y^2. \tag{IV.22}$$

From this we conclude

$$y = \sqrt{a^2 - a^2\frac{n^{n+1}}{[n+1]^{n+1}}} \tag{IV.23}$$

which is the value of y at the point **M**, where **CM** is the largest. If we make $n = 1$ to determine the value of the largest **CM** on the first curve, we will have

$$y = \sqrt{\frac{3}{4}a^2} \tag{IV.24}$$

which gives, by substituting this value of y into the equation,

$$x^4 = a^4 - a^2y^2, \tag{IV.25}$$

$$x = \sqrt{\frac{1}{2}a^2}. \tag{IV.26}$$

Therefore

$$\sqrt{x^2 + y^2} = \sqrt{\frac{5}{4}a^2} \tag{IV.27}$$

is the largest radius **CM**.

Extracted Records of The Royal Academy of Sciences, 18 May 1726

Messrs. Nicole and Pitot who were nominated to examine a memoir of geometry written by Mr. Clairaut the Younger, who is twelve and a half years of age,[15] on the properties and all the affections of the four geometric curves of third genre, which he found, and by the means by which he had an easier method for finding two or any number of mean proportionals which one seeks to find between two given lines, in having made their report, the company judged that this writing was deserving of much praise, and the new methods which will be very useful, and additionally the author, who being so young is quite knowledgeable, we cannot wait for all that he has to offer us. In witness whereof, I signed the present Certificate. In Paris 1 September 1726. FONTENELLE, Permanent Secretary of the Royal Academy of Sciences.

[15] 18 May was just after Clairaut's thirteenth birthday. Here, Nicole and Pitot must have been recognizing that the work was done when Clairaut was twelve and a half.