Jan Hudde's Second Letter: On Maxima and Minima

Translated into English, with a Brief Introduction* by

Daniel J. Curtin
Northern Kentucky University

Abstract. In 1658, Jan Hudde extended Descartes' fundamental idea for maxima and minima, namely that near the maximum value of a quantity the variable giving that quantity has two different values, but at the maximum these two values become one – algebraically a double root. He introduced more efficient ways of calculating double roots for polynomials and rational functions. His approach was the precursor of ours, equivalent to setting the formal derivative equal to zero, but his procedures were completely algebraic and based on a clever use of arithmetic progressions. Hudde also presented an early version of the Quotient Rule.

Hudde accomplished all of this in a letter to Frans van Schooten, which van Schooten published in his 1659 Latin edition of Descartes' *Geometria.* The letter is translated here from the photocopy of René Descartes, *Geometria,* with notes by Florimond de Beaune and Frans van Schooten, Fridericus Knoch, Frankfurt am Main, 1695, available at *Gallica* (Bibliothèque Nationale de France): http://gallica.bnf.fr/ark:/12148/bpt6k57484n. For Hudde's Second Letter, see Screens 523–532, which show pages 507–516. Photo images of a 1683 edition of the text, published in Amsterdam, are available at *e-rara* (ETH-Bibliothek, Zürich): http://dx.doi.org/10.3931/e-rara-24189. Images (pages) 507–516 contain Hudde's Second Letter.

Acknowledgment. The author wishes to express his gratitude to the reviewer who greatly improved the translation, correcting several errors and suggesting many improvements. Any remaining errors are, however, the author's own!

* For additional information on Hudde and the content of his letter, see the companion article, "Jan Hudde's Second Letter: On Maxima and Minima," in MAA *Convergence:* http://www.maa.org/publications/periodicals/convergence/jan-hudde-s-second-letter-on-maxima-and-minima

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The Second Letter:
On Maxima and Minima
by Jan Hudde

Most esteemed Sir,

As regards my method of maxima and minima I will attempt to describe it briefly, and to begin with I will prove this:

Theorem: If an equation has two equal roots and that equation is multiplied by an arbitrary arithmetic progression, namely the first term of the equation by the first term of the progression, the second term of the equation by the second term of the progression, and so on in succession, I say there is produced an equation in which only one of the roots mentioned remains.

To this end take any equation whatsoever in which x represents the unknown quantity as, for example, the equation[1]

$$x^3 + px^2 + qx + r = 0$$

and multiply it by $x^2 - 2yx + y^2 = 0$, that is, by an equation in which the two roots are equal. This yields the equation:

$$\left.\begin{array}{ll} x^2 - 2yx + y^2 \text{ times} & x^3 \\ x^2 - 2yx + y^2 \text{ times} & px^2 \\ x^2 - 2yx + y^2 \text{ times} & qx \\ x^2 - 2yx + y^2 \text{ times} & r \end{array}\right\} = 0\,,$$

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in which there will also be two equal roots, namely $x = y$ and for a second time $x = y$. Indeed if we had multiplied by $x^2 + 2yx + y^2 = 0$ we would have obtained two false[2] equal roots. If when we did the first multiplication we had put its value [x] for y we would have:

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[1] Following Descartes, Hudde actually wrote xx where we would write x^2. In most cases I have used the modern notation. The higher powers were written as we do.

[2] Hudde used Descartes' term "false" roots for negative roots.

$$\left.\begin{array}{ll} x^2-2xx+x^2 \text{ times} & x^3 \\ x^2-2xx+x^2 \text{ times} & px^2 \\ x^2-2xx+x^2 \text{ times} & qx \\ x^2-2xx+x^2 \text{ times} & r \end{array}\right\} = 0\,.$$

If then these four products or, which amounts to the same thing, $+1$, -2, $+1$ (when divided by x^2 – note that no change is made in the multipliers $x^3, px^2, qx, \& \; r$) are multiplied by an arithmetic progression, then the result of this multiplication will $= 0$.

For:

Multiplied	$+1$	-2	$+1$	Multiplied	$+1$	-2	± 1
by	a	$a+b$	$a+2b$	by	a	$a-b$	$a-2b$
makes	$a-2a-2b+a+2b$			makes	$a-2a+2b+a-2b$		
or	$+2a-2a+2b-2b=0$.			or	$2a-2a+2b-2b=0$.		

So far I have studied generally all equations having two equal roots, however they be given, whether they be missing certain terms or not, and whatever signs, + & –, they may have. It will be clear from our examination that only these numbers $+1$, -2, $+1$ matter, not the multipliers $x^3, px^2, qx, \& \; r$.

Likewise in regard to the arithmetic progression the matter remains quite general, since the two initial terms $a, a+b$ & $a, a-b$ are indeterminate. What remains, from just the consideration of the previous example, will become clear by comparing the two following multiplications. Clearly this gives:

$$\begin{array}{l} x^3+px^2+qx+r=0 \\ \underline{x^2-2xx+x^2=0} \end{array} \qquad \begin{array}{l} x^3+px^2+qx+r=0 \\ \underline{x^2-2yx+y^2=0} \end{array}$$

$$\left.\begin{array}{ll} x^2-2xx+x^2 \text{ times} & x^3 \\ x^2-2xx+x^2 \text{ times} & px^2 \\ x^2-2xx+x^2 \text{ times} & qx \\ x^2-2xx+x^2 \text{ times} & r \end{array}\right\} = 0 \qquad \left.\begin{array}{l} x^5-2yx^4+y^2x^3 \\ \quad +px^4-2pyx^3+py^2x^2 \\ \qquad\qquad +qx^3-2qyx^2+qy^2x \\ \qquad\qquad\qquad +rx^2-2ryx+ry^2 \end{array}\right\} = 0$$

Multiplied by $a, a\pm b, a\pm 2b, a\pm 3b, a\pm 4b, a\pm 5b$.

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Now since the products $x^5-2yx^4+y^2x^3$ and $x^2-2xx+x^2$ times x^3 turn out to be the same, so will $x^5-2yx^4+y^2x^3$ multiplied by $a, a\pm b, a\pm 2b$ be 0. And since $+px^4-2pyx^3+py^2x^2$ is the same as $x^2-2xx+x^2$ times px^2, then also

$px^4 - 2pyx^3 + py^2x^2$ multiplied by $a \pm b, a \pm 2b, a \pm 3b$ will be 0. (Indeed, as shown before, the first term of the progression may be taken at will.) And so on. From this it follows that the product of the entire equation with this series of proportions will = 0, but there will be only one value, $x = y$, of the two equal roots. Note that no use has been made of largeness or smallness or any other quality of things multiplied, so the proposition will be generally shown for any equation that has two equal roots.
From this follows:

If in some equation there are three equal roots and it is multiplied by any arithmetic progression, in the way that has been stated, there will remain two equal roots of the three. Then the product can be again multiplied by an arithmetic progression. For if the given equation has four equal roots and it is multiplied by an arithmetic progression there will remain three equal roots of the four, and so on. However many equal roots an equation has each of these multiplications will always reduce the number of equal roots by exactly one.

This having been proved, I will go on to my method of maxima and minima as follows.

Given any algebraic expression, which achieves either a maximum or, set the expression = z and after rearranging the terms, multiply by an arithmetic progression, in the way that has been stated.

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The result will be an equation that has a root in common with the given equation.

For the proof of this method it only remains to be shown that the first equation contains two equal roots. But this is so easy to prove that to pursue it further is nothing more than to waste toil and oil.[3]

And this is indeed my general method.

The special methods, which you have seen previously in several examples, reflect this. One may see how it can be done from the combined operations, using both methods.

1. *When the algebraic expression for which the maximum or minimum is sought contains but one unknown quantity and has no fraction in which the unknown quantity appears in the denominator*. I multiply each term by the degree[4] of the unknown quantity, neglecting all quantities in which the unknown does not appear and set the result = 0.

For example, let $3ax^3 - bx^3 - \frac{2b^2a}{3c}x + a^2b$ = some maximum value.

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[3] See the discussion in the Introduction of $f(x_0) = z$.

[4] Hudde called it "the number of the dimensions."

Multiply by $\quad \underline{3 \qquad 3 \qquad 1}$

to get $\quad 9ax^3 - 3bx^3 - \frac{2b^2a}{3c}x = 0$ or $9ax^2 - 3bx^2 - \frac{2b^2a}{3c} = 0$.

According to the general method this would be

$$3ax^3 - bx^3 * - \frac{2b^2a}{3c}x + a^2b = 0$$ [5]

$$-z$$

multiplied by the arithmetic progression $\quad \underline{3 \qquad 3 \quad 2 \qquad 1 \qquad 0}$

which gives, as before, $\quad 9ax^3 - 3bx^3 * - \frac{2b^2a}{3c}x = 0$

or $\quad 9ax^2 - 3bx^2 - \frac{2b^2a}{3c} = 0$.

2. *If the algebraic expression for which the maximum or minimum is sought contains a single unknown quantity but has several fractions in which the unknown quantity appears,* the established operation is possible, done like this: First I remove all known quantities. Then if the remaining

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quantities are not on the same denominator I reduce them under the same denominator. Once that is done, I consider the entire numerator with each member or separate part of the denominator (if it is made up of several parts) as a separate quantity achieving either the maximum or minimum and I multiply each member or separate part of the numerator by the degree[6] of the unknown of that term after I have subtracted from that number the degree of the unknown quantity that appears in the denominator. Then I multiply this by the denominator and set all the results = 0, as will become clearer from the following examples.

Example 1.

Let $\frac{4a^2b^3 + 5a^3x + x^5}{x^3} - ax + bx + ab$ be equal to some maximum.

After the known quantity ab is removed[7] and the remaining terms put on a common denominator, we have

$$\frac{4a^2b^3 + 5a^3x + x^5 - ax^4 + bx^4}{x^3}.$$

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[5] Hudde used * to represent a missing term, here x^2, following Descartes.

[6] Hudde called it "the number of the dimension."

[7] That is, the value of z doesn't affect where the maximum or minimum occurs.

Multiply the numerator by $\underline{-3, \quad -2, \quad +2, \quad +1, \quad +1}$
to get $-12a^2b^3 - 10a^3x + 2x^5 - ax^4 + bx^4$ multiplied by $x^3 = 0$.

But then, dividing by x^3, $-12a^2b^3 - 10a^3x + 2x^5 - ax^4 + bx^4 = 0$.

According to the general method, it is[8]

$$\begin{array}{l} \dfrac{4a^2b^3 + 5a^3x + x^5}{x^3} - ax + bx + ab = 0 \\ \qquad\qquad\qquad\qquad\qquad\qquad\quad -z\ . \end{array}$$

That is

$$\begin{array}{l} 4a^2b^3 + 5a^3x + x^5 - ax^4 + bx^4 + abx^3 = 0 \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad -z\ . \end{array}$$

Arranging the equation in order[9]:

$$\begin{array}{llllll} x^5 & -ax^4 + & abx^3 & * & +5a^3x & +4a^2b^3 = 0 \\ & +b & -z & & & \\ \hline +2, & +1, & 0, & -1, & -2, & -3 \end{array}$$

which is

$$\begin{array}{llllll} 2x^5 & -ax^4 & * & * & -10a^3x & -12a^2b^3 = 0 \\ & +b. & & & & \end{array}$$

Example 2.

Let $\dfrac{ba^2x + a^2x^2 - bx^3 - x^4}{ba^2 + x^3} - a + x$ be equal to some maximum.[10]

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Once the known quantity a has been removed and what is left put on a common divisor, there remains $\dfrac{2ba^2x + a^2x^2 - bx^3}{ba^2 + x^3}$.

Next for $\dfrac{\overset{+1,}{2ba^2x} + \overset{+2,}{a^2x^2} - \overset{+3}{bx^3}}{ba^2}$, I write $2ba^2x + 2a^2x^2 - 3bx^3$ times ba^2,

for $\dfrac{\overset{-2,}{2ba^2x} + \overset{-1,}{a^2x^2} - \overset{0}{bx^3}}{x^3}$,[11] I write $-4ba^2x - a^2x^2$ times x^3,

(and the total) = 0.

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[8] Note the indented $-z$ is aligned with the term ab of the same degree.

[9] By the alignment, Hudde implied the b in the second line represents bx^4 and z is zx^3 .

[10] In the Latin, x^3 and x^4 are mistyped as x_3 and x_4 .

[11] Again the Latin has x_3 instead of x^3 .

After dividing by a^2x there remains

$$\left.\begin{array}{ll} 2ba^2+2a^2x-3bx^2 & \text{times } b \\ -4bx-x^2 & \text{times } x^2 \end{array}\right\} =0$$

and so $-x^4-4bx^3-3b^2x^2+2a^2bx+2b^2a^2=0$.

Similarly,[12] if $\dfrac{2ba^2x+a^2x^2-bx^3+a^4}{4x^3+2bx^2-3a^2x-c^3}$ is equal to some maximum value:

For	$\begin{array}{c} -2, \quad -1, \quad 0, \quad -3 \\ \dfrac{2ba^2x+a^2x^2-bx^3+a^4}{4x^3} \end{array}$, I write	$-4ba^2x-a^2x^2-3a^4$ times $4x^3$,
for	$\begin{array}{c} -1, \quad 0, \quad +1, \quad -2 \\ \dfrac{2ba^2x+a^2x^2-bx^3+a^4}{2bx^2} \end{array}$,	$-2ba^2x-bx^3-2a^4$ times $2bx^2$,
for	$\begin{array}{c} 0, \quad +1, \quad +2, \quad -1 \\ \dfrac{2ba^2x+a^2x^2-bx^3+a^4}{-3a^2x} \end{array}$,	$+a^2x^2-2bx^3-a^4$ times $-3a^2x$,
for	$\begin{array}{c} 1, \quad 2, \quad 3, \quad 0 \\ \dfrac{2ba^2x+a^2x^2-bx^3+a^4}{-c^3} \end{array}$,	$+2ba^2x+2a^2x-3bx^3$ times $-c^3$,

(and the total) is $=0$.

By the general method,[13]

we have $\dfrac{2ba^2x+a^2x^2-bx^3}{ba^2+x^3}=z$

that is $2ba^2x+a^2x^2-bx^3=ba^2z+x^3z$

or[14] $\begin{array}{l} -bx^3+a^2x^2+2ba^2x-ba^2z=0 \\ -z \end{array}$

Arithmetic Progression

$$\begin{array}{l} \underline{\quad 3 \qquad 2 \qquad 1 \qquad 0 \quad} \\ -3bx^3+2a^2x^2+2ba^2x=0 \\ -3z. \end{array}$$

[12] This paragraph inserts another example.

[13] Hudde returned to the main example here.

[14] By the alignment, Hudde indicated the z in the second line represents zx^3.

So $\dfrac{-3bx^3+2a^2x^2+2ba^2x}{3x^3}=z,$

and hence $\dfrac{2ba^2x+a^2x^2-bx^3}{ba^2+x^3}=\dfrac{+2ba^2x+2a^2x^2-3bx^3}{3x^3}$

and as above $x^4+4bx^3+3b^2x^2-2a^2bx-2b^2a^2=0$.

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It is clear that these two special rules are based on that general method with respect to the progression 0, 1, 2, 3, 4, etc. multiplying, of course, the term in which the unknown term x does not appear by 0, where x had degree 1 by 1, and so on. But in the general case it is to be noted that the arithmetic progression may be chosen so that any selected term of the equation can be multiplied by 0. Thus the same value of z may be obtained as easily by one progression as by another. Thus in the previous example, where we multiplied by 3, 2, 1, 0, had we multiplied by 0, 1, 2, 3, we would have obtained

$$a^2x^2+4ba^2x-3ba^2z=0\text{, or }\frac{x^2+4bx}{3b}=z\,.$$

From this it is clear that the same quantity z, whether it be a maximum or minimum, and where x is assumed to be known, may be found and expressed in many different ways, from among which the easiest for the computation may be chosen. If instead it is assumed that z is known, then x can be found in just as many different ways. Further, taking both z & x as unknown, we may chose the equation with the simplest values, as in the above example between

$$z=\frac{-3bx^2+2a^2x+2ba^2}{3x^2}\text{ and }z=\frac{x^2+4bx}{3b}\,.$$

3. *If the algebraic expression for which the maximum or minimum is sought contains more than one unknown quantity*. I set those = z and by that equation and the others given, coming from the nature of the problem, I reduce all the equations to one, which must contain two unknowns, one of which is z. (This will always be the case when all the conditions of the problem are included, if the number of equations is the same as the number of unknowns less one, that is to say if one maximum or minimum value is sought among the infinite magnitudes, but not if there are infinitely many maxima). Since then only z needs to be considered for finding the maximum or minimum it is clear that to obtain this result the other unknown quantity must have two equal roots.

Let us suppose, for example, given the three equations, by which I determined the maximum width of the curve,[15] which appears on p. 498 of your *Mathematical Exercises* [Schooten], except the quantity

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[15] The curve is Descartes' folium. (See page 9 of this Translation.)

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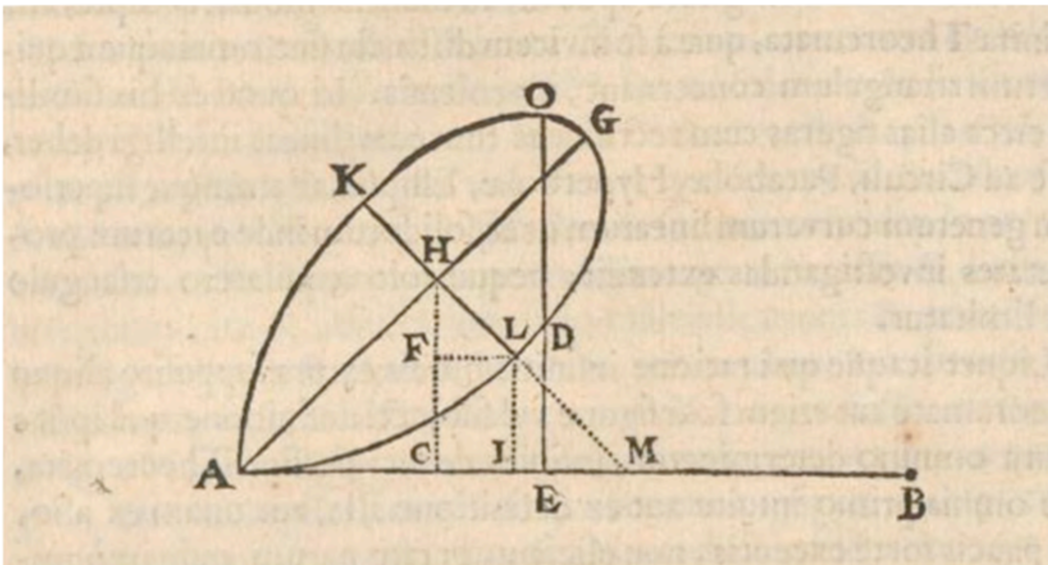
that is the maximum I call z and what is called z there, here I call v.

1st eq. $\quad y^3 - nyx + x^3 = 0$

2nd eq. $\quad v - x = y$

3rd eq. $\quad \frac{1}{2}v - y = z$ at maximum.



Frans van Schooten's diagram and derivation of the three equations from the cited p. 498, using Hudde's letters:

"Angle *GAB* is 45 degrees, which follows from the equation $y^3 - nyx + x^3 = 0$ since x and y appear symmetrically. Then letting $AI = x$, $IL = y$ and $AM = v$, we see that $IM = y$ and $AM = v = x + y$, that is $v - x = y$, which is the 2nd equation. Then since both AC and CH are $= \frac{1}{2}v$, so $FH = \frac{1}{2}v - y = z$, of all these lines this is to be the maximum,[16] This is the 3rd equation."

(Image used courtesy of ETH-Bibliothek, Alte und Seltene Drucke, ETH Zürich, Switzerland)

Substituting the value of y in the 2nd equation in place of y in the 1st and 3rd yields:

For the 1st eq. $\quad v^3 - 3v^2x + 3vx^2 = vnx - nx^2$

& for the 3rd eq. $\quad x = z + \frac{1}{2}v$.

[16] Since triangle *FHL* is isosceles, maximizing $z = FH$ will give the maximum of *HL* and thus of the width *KL*.

Substituting the value of x in this 3rd equation into this 1st equation gives

for the 1st eq. $\frac{1}{4}v^3 + 3vz^2 = \frac{1}{4}nv^2 - nz^2$

or $\frac{1}{4}v^3 - \frac{1}{4}nv^2 + 3z^2v + nz^2 = 0$.

And now, indeed, only this equation remains in which therefore, in order that the final condition of the problem be satisfied, that is, in order that it be determined in such a way that z will become a maximum, I multiply (just as is done here) the equation

$$\frac{1}{4}v^3 - \frac{1}{4}nv^2 + 3z^2v + nz^2 = 0$$

by the arithmetic progression $\quad 3, \quad 2, \quad 1, \quad 0$

and I get $\quad \frac{3}{4}v^3 - \frac{1}{2}nv^2 + 3z^2v + * \quad = 0$

or $\quad 3z^2 = \frac{1}{2}nv - \frac{3}{4}v^2$.

Then substituting the value of z^2 from this last equation in its place in the previous equation $\frac{1}{4}v^3 - \frac{1}{4}nv^2 + 3z^2v + nz^2 = 0$ the result is

$$\frac{1}{4}v^3 - \frac{1}{4}nv^2 + \frac{1}{2}nv^2 - \frac{3}{4}v^3 + \frac{1}{6}n^2v - \frac{1}{4}nv^2 = 0$$

or $\quad -\frac{1}{2}v^3 + \frac{1}{6}n^2v = 0$.

So $\quad v^2 = \frac{1}{3}n^2$.

If the arithmetic progression had been 0, 1, 2, 3, we would have found

$$3z^2 = \frac{nv^2}{8v + 4n} .$$

If 2, 1, 0, –1, then

$$3z^2 = \frac{\frac{3}{2}v^3 - \frac{3}{4}v^2n}{n} .$$

And whether the value of z^2 found from either of these equations be substituted in the previous equation $\frac{1}{4}v^3 - \frac{1}{4}nv^2 + 3vz^2 + nz^2 = 0$ or whether the one be equated to the other, using $\frac{1}{2}nv - \frac{3}{4}v^2 = \dfrac{nv^2}{8v + 4n}$, or $= \dfrac{\frac{3}{2}v^3 - \frac{3}{4}v^2n}{n}$, the result will always be $v^2 = \frac{1}{3}n^2$.

However, while these operations done in one way or another

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differ little from one another, it is often possible, as I mentioned above, that one way may be much lengthier or more difficult than another. In which case indeed it will be better to choose the more convenient way, which is easily determined.

Also note that the previous equation $\frac{1}{4}v^3 + 3vz^2 = \frac{1}{4}nv^2 - nz^2$ may be handled by the second of the preceding methods. Thus supposing $z^2 = \dfrac{\frac{1}{4}nv^2 - \frac{1}{4}v^3}{3v + n}$ and z = a maximum, then z^2 will have its maximum at the same time, thus

$$\text{dividing by } v^2 \;\; \frac{\left.\begin{array}{l} \frac{1}{4}nv^2 - \frac{2}{4}v^3 \text{ times } 3v \\ \frac{2}{4}nv^2 - \frac{3}{4}v^3 \text{ times } n \end{array}\right\} = 0}{\left.\begin{array}{r} \text{or } \frac{1}{4}n - \frac{2}{4}v \text{ times } 3v \\ \frac{2}{4}n - \frac{3}{4}v \text{ times n} \end{array}\right\} = 0},$$

that is,

$$\begin{array}{c} \left.\begin{array}{l} \frac{3}{4}nv - \frac{6}{4}v^2 \\ \frac{2}{4}n^2 - \frac{3}{4}nv \end{array}\right\} = 0 \\ \hline \text{or } \frac{2}{4}n^2 - \frac{6}{4}v^2 = 0 \\ \hline \text{i.e., } \frac{1}{3}n^2 = v^2 \end{array}$$

Now indeed in many cases when the final equation is reached it does not end up that the value of z itself, or of z^2 or z^3, etc., can be expressed in such terms that z does not appear, as it seemed the general case indicated in the working of this example.

Still, my Dear Sir, much remains to be said about this, but in order that my letter may not become a book, I stop, as if I would break the thread of my writing, especially since it is not difficult to obtain what might be desired from the preceding work. But lest what I believe you sought seem to be hidden from you, I add the outline of a work that I had prepared about this material 2 or 3 years ago for my own use, which you had recently looked over in passing as if through a screen. But in that work however are treated:

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I. *The Method of maxima and minima.*
The algebraic terms determining the maximum or minimum are investigated:

1. *Either with regards to our knowledge, when we may be sure its maximum can be found, if it has a maximum, or its minimum, if it has a minimum.*

   These algebraic terms contain:

   Either exactly one unknown quantity, having:
   1) *either no fraction in whose denominator the unknown quantity appears;*
   2) *or fractions in whose denominators the unknown quantity appears.*

   Or more than one unknown quantity, which has two cases:
   1) *either as many are given of these equations or are included in the nature of the problem as the number of unknown quantities, except one;*[17]

[17] The extra variable is z in Hudde's examples.

2) *or they are not the same (number of equations), or even none.*

2. *Or with regards to our ignorance* — that is, when we are not sure whether there is a maximum or a minimum, or whether there is neither — I examine them again either absolutely or relative to other problems.

II. *The use and usefulness of this (method),* which indeed is extended in depth and breadth especially to those problems whose equations would otherwise be difficult to resolve. The most outstanding example of this is *The determination of all equations*, which is certainly general and useful, is a corollary of this method.

Farewell, Dear Sir & accept my fond regards,

Your Most Obedient Servant,
Jan Hudde

Given in Amsterdam, 6 Cal February, 1658

THE END