Pathways from the Past

II: Using History to Teach Algebra

William P. Berlinghoff
Fernando Q. Gouvêa

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Contents

First Thoughts for Teachers

What's in this packet?

This packet contains a set of history-based student activities for learning about

The Symbols of Algebra
First-Degree Equations
Quadratic Equations
Cubic Equations and Complex Numbers

These activities are intended to supplement the contents of your regular math textbook. They are primarily mathematical; the settings are historical. Each chapter is largely independent of the others, to give you flexibility in fitting them into your lessons. Each section points out how its mathematical content fits into the math curriculum. It also indicates what connections might be made with topics in the history curriculum.

The activity sheets are designed as blackline masters. You can copy them directly for use with your students, or you can use their questions as prototypes for designing activities of your own.

Why use history?

To learn mathematics well at any level, students need to understand the questions before the answers will make any sense to them. To teach mathematics well at any level, you need to help your students see the underlying questions and thought patterns that knit the details together. Understanding a question often depends on knowing the history of an idea:

Where did it come from?

Why is or was it important?

Who wanted the answer and what did they want it for?

Each step in the development of mathematics builds on what has come before. The things that students need to know now come from questions that needed to be answered in the past. Yesterday's questions can help students understand and use today's tools to deal with tomorrow's problems.

Most students are naturally curious about where things come from. That curiosity needs to be encouraged and nurtured. It is a powerful but fragile motivator. With some care, you can use it to lead your students to go beyond merely being able to do things. Every bit of mathematics makes sense, but many students never really get to the point where mathematical processes become meaningful. The story of why and how those processes were developed can help them learn what they need to know.

How can I use history to do this?

There are several ways to use history in the classroom. The most common way to do it is simply to use *stories*. If chosen carefully, a story about a historical person or event can help students understand and remember a mathematical idea. The main drawback of using stories is that often they are only distantly connected to the mathematics.

This packet is devoted primarily to presenting the historical background for student activities, so stories do not play a major role in the material that follows. Instead, we focus on the following ways to use history:

Overview — It is all too common for students to regard school mathematics as a random collection of unrelated bits of information. But that is not how mathematics actually gets created. People do things for a reason, and their work typically builds on previous work. Historical information helps students to see this "bigger picture." It also often explains why certain ideas were developed. Many crucial insights come from crossing boundaries and making connections between subjects. Part of the big picture is the fact that these links between different parts of mathematics exist, and paying attention to their history is a way of making students aware of them.

Context — Mathematics is a cultural product, created by people in a particular time and place and often affected by that context. Knowing more about this helps us understand how mathematics fits in with other human activities. For instance, the idea that numbers originally may have been developed to allow governments to keep track of data such as food production embeds arithmetic in a meaningful context right from the beginning. It also makes us think of the roles mathematics still plays in society. Collecting statistical data, for example, is something that governments still do!

Depth — Knowing the history of an idea usually leads to deeper understanding. For example, long after the basic ideas about negative numbers were discovered, mathematicians still found them difficult to deal with. They understood the formal rules for them, but they had trouble with the concept itself. Because the concept was troublesome, they did not see how to interpret those rules in a meaningful way. Learning about their difficulties helps us understand (and empathize with) the difficulties students might have. Knowing how such difficulties were resolved historically can also help students overcome these roadblocks for themselves.

Activities — History is a rich source of student activities. It can be as simple as asking students to research the life of a mathematician, or as elaborate as a project that seeks to lead students to reconstruct the historical path that led to a mathematical breakthrough. The activities in the worksheets of this packet ask students to do specific things that will deepen their understanding of particular mathematical ideas and help them practice their skills. The Teacher Notes occasionally suggest some broader questions and projects that might be appropriate from time to time, at least for some students.

What if I don't know much history myself?

We'll help. To begin with, each chapter in this booklet contains a summary of the historical background for its topic. Then the notes for each activity sheet connect the relevant history with detailed solutions for all of the questions. The references to the bibliography at the end of the booklet provide specific sources for more detail, in the event you choose to pursue some of these ideas further.

For an easily readable, compact, inexpensive source of further background information, we unabashedly suggest our own book, *Math through the Ages: A Gentle History for Teachers and Others* (Oxton House Publishers, 2002). That book also contains a section describing "What to Read Next" and an extensive bibliography for anyone who wants to do serious historical research about a topic or a person.

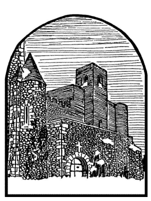
What do I really need to know first?

Not much — just a few small pieces of the Big Picture of the past several thousand years. Most of the algebra we now learn in school comes from a tradition that began in India and the medieval Islamic empire. Later this tradition migrated to late-Medieval and Renaissance Europe, and eventually became mathematics as it is now understood throughout the world. Some other cultures (Chinese, for example) developed their own independent mathematical traditions, but they have had relatively little influence on the substance of the mathematics that we now teach. That independence of cultures is one thing to keep in mind as you think about the history of mathematics. Here are some other major historical reference points that your students might know from their history classes.

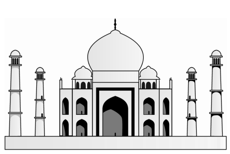
Ancient **Mesopotamia**, the region between the Tigris and Euphrates Rivers in what is now Iraq, is sometimes called the "cradle of civilization." If your students are studying ancient history, they will probably recognize the name of Hammurabi, a prominent ruler of the Babylonian Empire of the 17th century BCE. This was several thousand years after the beginnings of civilized society in that region, but it provides a convenient reference point. By Hammurabi's time, both writing and numeration were well developed tools for communication and commerce. With a centralized government came a professional class of scribes who developed both literature and mathematics.

Ancient **Egypt** also had a well developed civilization by that time. It had developed around the Nile Valley in northern Africa. By about 3000 BCE, the country began to be unified under a single ruler (a Pharaoh). The pyramids date back to this millennium (3000–2000 BCE). Our main source of information about Egyptian mathematics comes from an artifact of a later time, the Ahmes Papyrus (or Rhind Papyrus) of about 1650 BCE. Egyptian mathematics seems to have been built around problems expressed in everyday contexts such as weighing stones, making baskets, and supplying food to workers.

The Ancient **Greek civilization** dates from about the 6th century BCE to the Roman conquest, often identified with the Battle of Corinth in 146 BCE. In the Roman period that followed, however, Greek culture remained influential and mathematics in the Greek tradition continued to be developed, so that the small but influential stream of "Greek mathematics" actually spans about 1,000 years. In the 5th century BCE, Athens was the center of an intellectual and artistic culture. It was the era of dramatists Sophocles and Euripides, philosophers Socrates and Plato, and other prominent artists and intellectuals. When Alexander the Great's conquests spread Greek culture to the Near East and north Africa in the 4th century BCE, Alexandria in northern Egypt — named after the great conqueror — became another center of Greek learning. Most of the Greek mathematics we know about comes from the period after Alexander the Great, and a lot of it seems to be connected to Alexandria. It is associated with names such as Euclid, who lived in Alexandria, and Archimedes, who lived in Syracuse (in Sicily).



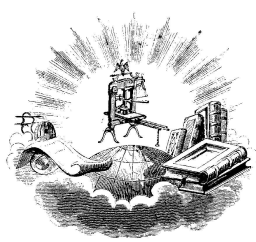
The **fall of Rome** brought with it the political and social fragmentation of the **Middle Ages**. Between the 5th and 10th centuries CE, European scientific scholarship was essentially dormant, partially preserved but not materially enhanced by the Christian Church and its monasteries. During that time, two other cultures that would influence Western mathematics were developing independently.



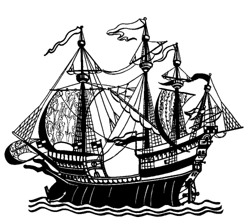
One was in **India**, where scholars were making great strides in astronomy, mathematics, and other intellectual pursuits. However, by the communication and travel standards of that time, India was very far from Europe, so almost none of this work found its way directly into the European tradition. That cultural link was provided by the Arabs, in a land geographically between India and Europe.

The rise of Islam began with the Hejira in 622 CE. A hundred years later, a loosely knit **Islamic Empire** stretched from India in the East to Spain in the West. It encompassed all of the African coast of the Mediterranean Sea, the entire Arabian Peninsula, and all of the Middle East, north to the Black and Caspian Seas. The last remnant of Greek culture was the Byzantine empire, centered in modern-day Turkey and constantly at war with the powers to the East. The caliphs (rulers) of 9th century Baghdad actively fostered the study of mathematics and science, drawing on both Greek and Indian sources. The mathematicians and scholars came from many places and different cultures, but the all wrote in the common scholarly language throughout the Islamic Empire, Arabic.

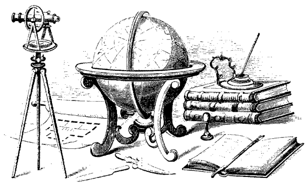
European and Arabic powers came into close contact in Spain, in the Baltic region, along the Mediterranean, and in the many conflicts known as the Crusades. All of this opened the way to increased **commerce between East and West**. As goods and money flowed, so did ideas. By the 10th century, the scientific and mathematical advances of the Arabs were beginning to make their way into the European scholarly tradition as the Arabic manuscripts were gradually translated into Latin.



The exchange of ideas throughout Europe was aided by the use of **Latin** as the common language of scholarship. However, it was hampered by the fact that all copies of documents had to be transcribed by hand. That changed in 1440 with the invention of **movable-type printing** by Johannes Gutenberg in Germany. The famous Gutenberg Bible of 1454 was the first book printed by this method. As printing spread, so did ideas. This invention was possibly the most influential technological breakthrough of the 15th century.



The **European Renaissance** of the 14th – 16th centuries awakened a renewed interest in science and mathematics, along with many extraordinary achievements in art, literature, and philosophy. Advances in shipbuilding and in craftsmanship of all kinds led to a wider range of commercial activity and exploration. This, in turn, called for more refined technological tools. It was the era of Christopher Columbus, Vasco da Gama, Ferdinand Magellan, John Cabot, and other seagoing adventurers. These activities required advances in such things as navigation, trigonometry, astronomy, and clock-making. Merchants learned arithmetic and developed double-entry bookkeeping.



The 17th and 18th centuries were a time of sweeping philosophical and political change in Western Europe. The rationalist philosophers of the time described it as the **Age of Reason** or the **Enlightenment**. This was the time of Galileo and Newton, who used mathematics to develop their scientific ideas. Many of the philosophers of the time, such as René Descartes in France and Gottfried Wilhelm Leibniz in Germany, were also mathematicians. Across the English Channel, Thomas Hobbes and John Locke were developing empirical philosophies consistent with the growing enthusiasm for experimental science. European influence was spreading all over the world, from Asia to the Americas.

The historical scope of this booklet and its activity sheets lies almost entirely within the boundaries of these milestones. By the 18th century, most of the algebra we now teach in school was already known to those very few who were educated beyond the basics.

1

Writing Algebra

Mathematical Focus	**Using Algebraic Symbols**
Historical Connections	Europe in the 12th to 17th centuries

When you think of *algebra*, do you think of equations made up of x's and y's strung together with numbers and symbols? Many people do. In fact, many people think of algebra *only* as a bunch of rules for manipulating symbols that have something to do with numbers. There's some truth in that. But describing algebra solely in terms of its symbols is like describing a car by its color and body style. What you see is *not* all you get. In fact, most of what makes algebra run is "under the hood" of its symbolism. Nevertheless, the symbols are important. Just as an automobile's body styling can affect its performance and value, so does the symbolic representation of algebra affect its power and usefulness.

Clear, precise notation has long been recognized as a valuable ingredient in the progress of mathematical ideas. We write clearly *in order to* think clearly. But many students struggle with the symbols of math. Just seeing an equation can make their eyes glaze over and their minds go numb. Sometimes that's because they don't understand what the symbols mean, but often the difficulty is more subtle.

Symbolic algebra is a language of abbreviations. It lets you pack a lot of information into a small space. Some students who understand the individual symbols may not be able to absorb the meaning of an equation because they can't easily process so much information in such compact form. This is particularly true of students who read quickly and tend to be impatient. For them, the brevity of the notation actually *inhibits* its clarity.

The activities in this chapter focus on understanding algebraic notation. Working through the evolution of some of our most basic symbols will give students a clearer picture of what they mean. The unfamiliar notations used in previous centuries will slow down even the most impatient students without making them feel that they "ought to" be moving along faster.

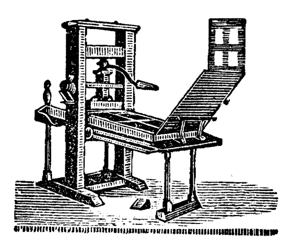
Working with older algebraic notation (or sometimes, the lack of it) can also show students the value of a well-organized notational system. Older notations may be easier to decipher because they are closer to spoken language, but they may also be very difficult to work with. And studying some of the conceptual missteps that mathematicians made along the way will underscore some of the most important features of our current way of symbolizing algebraic ideas.

[*Acknowledgement:* Much of the information in this section is drawn from Florian Cajori's encyclopedic book, *A History of Mathematical Notations*, Vol. 1, which is [4] in the bibliography.]

Sheet 1-1: Symbols of Arithmetic

• MAIN FEATURE •
Writing operations on numbers

The symbols of arithmetic have become universal, far more commonly understood than the letters of any alphabet or the abbreviations of any language. But that hasn't always been so. The ancient Greeks and early Arab scholars didn't use symbols for arithmetic operations or relations; they wrote out their problems and solutions in words. In fact, arithmetic and algebra were written entirely in words by most people for many centuries, right through the Middle Ages. Arithmetic symbols arose as shorthand early in the Renaissance, with little consistency from person to person or from country to country. With the invention of movable-type printing in the 15th century, books began to exhibit a little more consistency, but it was a long time before today's symbols became a common part of written arithmetic.



Your students might better understand the motivation for efficient symbolism if you tell them a bit about movable-type printing. The process is so far removed from our modern business machines that they may not appreciate how hard it was to produce even a few pages of printed matter. From Gutenberg's invention of the printing press in the mid-15th century until the late 19th century, typesetting was done by hand. Each piece of type was a small chunk of lead with a letter on it.

The printer would place one piece of type for each letter or symbol into a "galley," a shallow tray designed to hold rows of type. Ink was rolled onto the galley and paper pressed against it. For this to work, the type has to be placed into the tray *backwards*. The printing plate for each page of any book or paper was made this way. No wonder people looked for shorter ways of expressing their ideas!

Most printers had molds from which they could make as many copies of a letter as they needed, but if they needed an unusual symbol, they had to carve a new mold by hand. Positioning a symbol correctly was also tricky, and printers often got it wrong. The printer was much happier, and the results were better, if you used symbols already on hand, so it was good to standardize symbolism.

In this activity sheet, students work with various arithmetic symbols to see how the ones we use today evolved. The unfamiliarity of the symbols encourages students to think about how and why notation is used. It also paves the way for later work with symbols for variables and their powers.

Solutions

1. $(4 + 7) - 9 = 2$. This opinion question prompts students to think about the convenience of symbols. Most students probably will choose "in symbols" as clearer. Reasons given by students who choose "in words" instead, may provide insights that will help you resolve difficulties those students might be having with symbolic algebra.

2. The only tricky part of these questions is the grouping. The exact words may vary a bit from the answers given here, but check that your students' words accurately reflect the way the numbers are grouped in the equations. The form of the first equation mimics the example of #1.

   (a) When 7 is subtracted from the sum of 5 and 6, the result is 4.

   (b) 24 minus the sum of 9 and 6 is 10 minus 1.

3. Johannes Müller (1436–1476) was called Regiomontanus (a Latin version of "from Königsberg," his birthplace). He is best known for *De Triangulis Omnimodis* ("On All Sorts of Triangles"), one of the first books devoted solely to trigonometry. Like most scholars of his time, he wrote in Latin. The origin of his minus symbol is not known, but may be a script abbreviation for a Latin word for subtraction. Regiomontanus did not use parentheses or other grouping symbols. It was assumed that the calculations would be done in order from left to right.

   (a) 5 *et* 6 $\overset{\frown}{\iota 9}$ 7 — 4

   (b) 12 $\overset{\frown}{\iota 9}$ 5 — 4 *et* 3

   (c) $17 - 6 + 4 = 12 + 8 - 5$.

4. As the Italian merchants of the 15th century developed their businesses, they had more and more need of calculation. Mathematicians tried to meet this need by writing books on arithmetic and algebra in everyday Italian. The high point of this tradition was Luca Pacioli's *Summa de Aritmetica, Geometria, Proportione e Proportionalita* ("Summary of Arithmetic, Geometry, Ratios and Proportionality") of 1494, a huge compendium of practical mathematics, from arithmetic to bookkeeping. Movable-type printing had just been developed in Germany and was flourishing in Italy. Pacioli's *Summa* was one of the first mathematics books to be printed.

   (a) $12\ \tilde{m}\ 5 \text{—} 4\ \tilde{p}\ 3$

   (b) $12 + 28 - 9 = 25 + 6$

5. (a) $(9 - 5) - (3 + 4) = 4 + 7$

   (b) $27 + (13 - 6) - (11 + 4) - (3 + 6) = 10$

   (c) True: $120 - (46 - 17) = 120 - 46 + 17$. As in any true/false question, it is useful here to ask students to defend their choice.

6. Recorde's notation was not immediately popular. Many European writers preferred to use $\tilde{p}$ and $\tilde{m}$ for plus and minus. His equality symbol, which he justified by saying that "no two things can be more equal" [than parallel segments of the same length], didn't appear in print again for more than half a century. Meanwhile, = was being used for other things by some influential writers. For instance, it was used for the absolute value of the difference in a 1646 edition of François Viète's collected works.

   You might ask your students if they know what a *whetstone* is. (It's a stone for sharpening knives, scythes, and the like. "Whet" means "sharpen.") The word "wit" (which Recorde writes as "witte") today tends to suggest humor, but at the time referred to intelligence in general. So the title of his book is one of the first statements that "algebra makes you smarter."

   (a) 68 —— 23 ═══ 40 —+— 5.

   (b) $17 - 5 = 9 + \mathbf{3}$

7. In his influential book, *Clavis Mathematicae* (The Key of Mathematics), British mathematician William Oughtred wrote that the presentation of mathematics in symbols "neither racks the memory with multiplicity of words, nor charges the phantasie [imagination] with comparing and laying things together; but plainly presents to the eye the whole course and process of every operation and argumentation."[1] His use of +, −, and = influenced their eventual adoption as standard notation, but not for quite a while.

[1] From William Oughtred's *The Key of the Mathematicks* (London, 1647), as quoted in [4], p. 199, adjusted for modern spelling.

(a) $24 - (5 + 7) - 2 + (8 - 3) = \mathbf{15}$

(b) True: $(18 + 10) - 7 = 18 + (10 - 7) = 21$

(c) False: $(18 - 10) + 7 = 15$, but $18 - (10 + 7) = 1$.

8. René Descartes was one of the most prominent mathematicians of the 17th century. His writings influenced mathematics and science throughout Europe. In particular, his algebraic notation spread rapidly, simplifying and regularizing what had been a hodgepodge of individual systems. As we shall see later, much of his notation was very much like what we use today, with the glaring exception of the equal sign. Because of his influence, that equal sign was used in some parts of Europe, particularly in France and Holland, until the early 18th century.

   (a) $18 - (5 + 6) + 3 - 2 = \mathbf{8}$

   (b) $37 + 5 - (14 + 6) - (8 + 4) = \mathbf{10}$

   (c) True. $20 - 3 - (12 + 4) = 1$

   (d) True. $15 - (9 + 5) = 15 - 9 - 5$

   (e) False. $(40 - 25) - 10 \neq 40 - (25 - 10)$. (Subtraction is not associative.)

9. The point of this question is to get students thinking about what was happening in the world outside of mathematics at the time of Descartes, Oughtred, and their contemporaries. The first half of the 17th century (1600–1650) was a time when major European nations — most notably England — were beginning to establish permanent colonies in the New World. Listed below are some particulars that students might cite.

   - The British established major colonies in what are now these states:
     - Virginia (Jamestown 1607)
     - Massachusetts (Plymouth 1620)
     - Maryland (1633)
     - Connecticut and Rhode Island (1636)
     - Delaware and New Hampshire (1638)
   - The Dutch settled Manhattan Island in New Amsterdam (now New York) in 1626.
   - The Spanish were strengthening their outposts in Florida, which had been settled in the 16th century.
   - The French founded colonial cities in Canada (e.g., Port Royal 1605, Quebec 1608, Montreal 1642).
   - The British invaded parts of French Canada.

Many other good answers are possible, of course, depending on your students' knowledge of early American history. Whatever events are cited, students should recognize that North America was pretty primitive then. It would be some time before the sophisticated mathematical ideas of Europe gained a firm foothold in the New World.

Sheet 1-2: Algebra in Italy, 1200–1550

• MAIN FEATURE •
Symbols for the unknown

An algebra problem, no matter how it's written, is a question about finding an unknown quantity from known ones. Here's a simple example:

> Twice the square of a thing is equal to five more than
> three times the thing. What is the thing?

Despite the absence of symbols, this is an algebra question. The word "thing" was a respectable algebraic term for a very long time. In the 9th century, Muḥammad Ibn Mūsa al-Khwārizmī (whose book title, *al-jabr w'al muqābala*, is the source of the word "algebra") used the word *shai* to mean an unknown quantity. When his books were translated into Latin, this word became *res*, which means "thing." For instance, John of Seville's 12th-century elaboration of al-Khwārizmī's arithmetic contains this question, which begins "Quaeritur ergo, quae res...":[2]

> It is asked, therefore, what thing together with 10 of its roots or what is
> the same, ten times the root obtained from it, yields 39.
> (In modern notation, this would be $x + 10\sqrt{x} = 39$ or $x^2 + 10x = 39$.)

Some Latin texts used *causa* for al-Khwārizmī's *shai*, and when these books were translated into Italian, *causa* became *cosa*, the ordinary Italian word for "thing." As other mathematicians studied these Latin and Italian texts, the word for the unknown became *Coss* in German. The English picked up on this and called the study of questions involving unknown numbers "the Cossic Art" — literally, "the Art of Things."

Good mathematical notation is far more than efficient shorthand. It is a universal language that clarifies ideas, reveals patterns, and suggests generalizations.

[2]See p. 336 of [4] for both the original Latin and this translation.

Without consistent symbolism, early algebra was indeed an art that depended heavily on the skill of its individual practitioners. Just as standardization of parts was a critical step in the mass production of Henry Ford's automobiles, so the standardization of notation was a critical step in the progress of algebra.

The development of current algebraic notation has been long, slow, and sometimes convoluted. Like most of our familiar algebraic symbols, the x and other letters we now use to represent unknown numbers are relative newcomers to the "art." Their story involves a big step forward in the evolution of mathematics. Exploring some of the steps in that story can help students understand the meanings of the equations and formulas we write today. In this activity sheet and the next, students get to work with the symbol systems used by some of the most famous mathematicians of the 16th, 17th, and 18th centuries. As they do, they will

- see what modern algebraic symbols mean,
- get practice solving equations and deciphering formulas, and
- gain a deeper understanding of algebraic manipulation.

The novelty of these unfamiliar settings allows you to review and reinforce fundamental algebraic ideas in an engaging context.

Solutions

1. Leonardo of Pisa is perhaps better known as Fibonacci. His most famous book, *Liber Abbaci* ("Book of Calculation") was published in 1202. Pisa of the late 12th and early 13th centuries was one of the major city-states in northwestern Italy, frequently at war with its northern neighbor, Genoa. In the broader context of European history, this was the era of the Magna Carta in England and of the Fourth Crusade, which, instead of fighting the Muslims in Jerusalem, ended up sacking the Christian city of Constantinople.

   (a) Leonardo would have looked only for positive solutions, since negatives were not regarded as legitimate numbers at that time. If 5 more than the square is 14, then the square is 9, so the "thing" is 3.

   (b) $x^2 + 5 = 14;\ \ x = ?$

   (c) Three less than the square of a thing is equal to two more than four times the thing. What is the thing?

   (d) If your students have not done much algebra yet, you can skip this part. The equation can be rewritten as $x^2 - 4x - 5 = 0$ or $(x - 5)(x + 1) = 0$. Leonardo would have accepted 5 as a solution, but not -1.

2. (a) $2x^2 + 4 = 6x$

   (b) $10x = x^3 + 3$

(c) $(x-1)^2 = 2(x+3)$

(d) These can be done in a variety of ways. They have been chosen carefully to have easily accessible solutions. Students can find solutions by looking at the numerical patterns, perhaps with the aid of a calculator. This kind of exploration helps to break students of the counter-productive tendency to think that there is only one "right" way to solve an algebra problem.

a: Start by writing the left and right sides of the original equation for $x = 1, 2, 3, 4, \ldots$. The left side is $6, 12, 22, 36, \ldots$; the right side is $6, 12, 18, 24, \ldots$. Since the first two values match, $x = 1$ and $x = 2$ are solutions.

b: Using the pattern-matching strategy of part (a), the left values for $x = 1, 2, 3, 4, \ldots$ are $10, 20, 30, 40, \ldots$, and the right values are $4, 11, 30, 67, \ldots$. This shows that 3 is a thing that works.

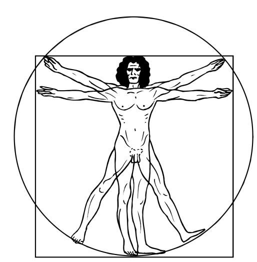
c: This, too, can be done by pattern-matching. The left side is 0, 1, 4, 9, 16, 25, ... and the right side is 8, 10, 12, 14, 16, 18, ... They agree for $x = 5$, so 5 is a thing that works.
(Note: In much of this early work on algebra, there seems to have been little concern about whether one had found *a* solution or *all* the solutions.)

3. Wording may vary, of course. The key idea here is to avoid using symbols, particularly for the unknown and its powers. The numbers may be written in symbols.

(a) Taking a thing from 25 yields seven more than four things, or seven more than four times a thing is equal to 25 minus the thing.

(b) One less than twice the square of a thing is equal to 3 more than 5 times the thing.

(c) The cube of a thing minus twice its square is equal to 9 more than 10 times the thing.



As Europe emerged from the Middle Ages into the Renaissance, the cities of Europe developed as centers of commerce and culture. This was the age of Columbus, Copernicus, Michelangelo, and da Vinci. (The graphic here shows a famous da Vinci proportionality sketch known as *Vitruvian Man.*) The location of many Italian cities along the seas of the Mediterranean made them natural centers of commerical activity in that age of seagoing travel and trade. The invention of movable-type printing in the mid-15th century was a vital force for the spread of ideas and learning throughout Europe. With the increase in trade, banking, and other forms of business came the need for reliable mathematical methods to handle more sophisticated tasks, from bookkeeping to science. Luca Pacioli's *Summa de Aritmetica, Geometria, Proportione e Proportionalita* was the most widely known math book of that time. It was a huge compendium of practical mathematics, from everyday arithmetic to double-entry bookkeeping. One of the first mathematics books to be printed, it circulated widely and became the basis for much later work in algebra.

About Words: The symbols in the next problem need some explanation. Pacioli was writing in Italian, a language close to Latin. When al-Khowarizmi's work was translated into Latin, the Arabic for "thing" became *causa* and the Italians made it *cosa*. Latin for the cube of the thing was *cubus*, which became *cubo* in Italian. The word for the square of the thing is less obvious. The Arabs used *mal*, a word meaning "property." If you think of property in terms of land owned, then the connection to area and hence to square measurement becomes plausible. The Latin word for property or wealth is *census*[3], which became *censo* in Italian.

4. (a) *cu* (*cubo*) is the cube of the thing; that is, x^3. As in Sheet 1-1, p̃ is + (*piú*) and m̃ is $-$ (*meno*). The long dash stands for =. The coefficient 1 was used for single copies of powers of the thing.
   (b) The dots are merely to the separate the symbols to avoid ambiguity. This was particularly important before the days of movable-type printing, when work was done and transcribed by hand. It carried over to the early years of printed books.
   (c) $1 + 7x^2 = 2x^3 - x$.
   (d) *5.cu.p̃.3.co.* —— *1.ce.m̃.2.*

The questions of #5 and #6 invite students to do some basic manipulative algebra in this unfamiliar context.

5. The symbol ℞, as well as much of Pacioli's other notation, was used by influential Italian mathematicians, including Girolamo Cardano, for much of the 16th century.
   (a) $\sqrt{9+\sqrt{49}} = 4$. Yes, it is correct: $\sqrt{9+7} = \sqrt{16} = 4$.
   (b) $\sqrt{4x^2 - 12x + 9} = 2x - 3$. Yes, it is correct, in the sense that this is one root of the quadratic. That is, $(2x-3)^2 = 4x^2 - 12x + 9$.
   (c) *℞.9.p̃.℞.16.* —— *℞v.50.m̃.1.* It is correct: $3 + 4 = 7$.
   (d) *℞v.1.ce.m̃.4.co.p̃.4.* —— *1.co.m̃.℞.4.* It is correct: $(x-2)^2 = x^2 - 4x + 4$.

[3]The Roman census catalogued the wealth of the Empire.

6. Students will probably need some scrap paper for these questions. If they translate the problems into modern notation in order to solve them (as they probably will), that will help them appreciate the convenience of the notation we use today.

   (a) $2x^3 - 3x^2 - \sqrt{4}x = x(2x+1)(x-2)$

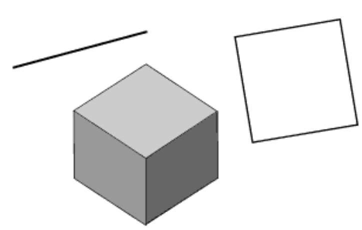
   (b) In modern notation, this is $(3x^2+5)(4x-1) = 12x^3 - 3x^2 + 20x - 5$. In Pacioli's notation, it is *12.cu.m̃.3.ce.p̃.20.co.m̃.5.*

   (c) $9x^2+2x-16 = \sqrt{4x^2}$, so $9x^2+2x-16 = 2x$, or $9x^2-16 = 0$. $9x^2-16$ is a difference of two squares, which factors into $(3x+4)(3x-4)$. Setting each equal to 0 yields $x = \pm\frac{4}{3}$. Thus, a positive solution is $\frac{4}{3}$.

Sheet 1-3: Germany and France, 1450–1600

• MAIN FEATURE •
Higher powers of the unknown

This activity sheet focuses on a notational habit that delayed the conceptual development of algebra. The way people of the 16th century thought about algebraic problems was restricted by the way they wrote the symbols! As we saw in the previous activity sheet, Italian mathematicians of the 15th and 16th centuries used different symbols for the unknown and its powers. That was true elsewhere in Europe, too. This convention may well have been due to the influence of early Greek mathematics. The Greeks' conception of number was tied closely to geometry. Numbers that represented lengths or areas or volumes were thought of as different *kinds* of things, and that carried over to European algebra. Just as we would not add 1 ft. + 2 sq.ft. + 3 cu.ft. to get 6 (of what?), the early algebraists treated an unknown, its square, and its cube as different kinds of numbers. Those words, which we still use, reflect that tradition. However, the notational complexity of maintaining that separation inhibited the scope and flexibility of algebraic thinking. For example, it obscured the fact that x^3 can be thought of as the product xxx. It took several centuries for European algebra to gain the flexibility it has today.



The questions on this sheet highlight the difficulty caused by trying to keep this separation and the first steps in breaking away from it. As before, the mathematical benefit to students comes from working through some fundamental algebraic processes in these unusual settings. Thinking about how to express familiar ideas in unfamiliar ways encourages a deeper understanding of the ideas our own algebraic symbols express. Along the way, there is also some practice with factoring common quadratics, including the difference of two squares. If your students have not yet studied these things, you can simply give them the factorizations.

Solutions

1. Unlike the Italians and many other European mathematicians, the Germans were using $+$ and $-$ for addition and subtraction by the end of the 15th century. These signs first appeared in 1489 in a book on commercial arithmetic by Johann Widman. German algebraists adopted them in the 16th century. The notation in Christoff Rudolff's *Coss* of 1525 (which has an impossibly long formal title) is typical of German algebra at the time. In particular, his notation for the powers of the unknown were in general use then.

   (a) ꞓe represents x^3. The word (from Latin) was *cubus*; this symbol is just a small c with an extra curl.

   ʒ represents x^2. The German word was *zensus*; this symbol is its first letter, a lowercase script z.

   ℛ represents x, the unknown. The word used was *radix*, which means "root," in the sense that it was the (square) root of the *zensus*.

   (b) This question exemplifies the answer to part (c) that follows. It leads students to notice from the example at the top of the sheet that Rudolff used dots to indicate when a square root applied to more than one term. Therefore, this equation translates as $\sqrt{x^2+2x+1} = \sqrt{x^2}+1$. That is, $\sqrt{(x+1)^2} = x+1$ (choosing the positive root for both). The factorization of the quadratic on the left is a common form that students should either recognize immediately or be able to find easily.

   (c) As observed in part (b), Rudolff (and others) used a dot after the square root sign, and often used a second dot at the end of the scope of the radical, unless that was obvious.

   (d) $5x^2 - 20 = \sqrt{x^2+4x+4}$, so $5(x^2-4) = \sqrt{(x+2)^2} = x+2$. (Rudolff would have considered only the positive square root.) Now, x^2-4 is the difference of two squares, so it factors into $(x+2)(x-2)$. Therefore the equation becomes $5(x+2)(x-2) = x+2$. Cancel $x+2$ (which is OK except for $x=-2$) to get $5x-10=1$. Therefore, $x = \frac{11}{5}$.

   (e) 3ꞓe $+$ 5ʒ $-\sqrt{7}$ *aequ.* 4ℛ $+ \sqrt{.}$ꞓe $-$ ℛ $+ 6$.

2. This is an exercise in remembering the laws of exponents and seeing patterns. Rudolff used a multiplicative approach for representing higher power of the unknown. Part (d) asks students to describe this pattern; parts (a)–(c) lead them to it.

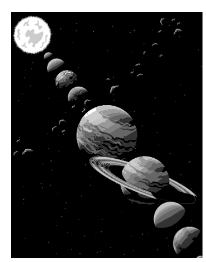
   (a) Since this symbol represents x^6, it represents $(x^2)^3$ (perhaps the most obvious translation of the symbols), as well as $(x^3)^2$, which is $x^3 \cdot x^3$. It does *not* represent $x^2 \cdot x^3$ or $x^3 \cdot x^2$ because they equal x^5.

   (b) By the chart, this symbol represents x^8. Therefore, it also represents $(x^4)^2$ (product of exponents), but not $x^4 \cdot x^2$ (sum of exponents).

   (c) To be consistent with the pattern in his table on the activity sheet, he would have used a combination of symbols for powers whose product is the desired power. x^{10} would be ʒß, or perhaps ßʒ. (The first of these is consistent with his representation of x^6.) x^{12} would be ʒʒce.

   (d) When possible, the symbol for a higher power is formed by putting together lower-power symbols so that the product of their "sizes" is the desired power.

   (e) 11 is prime, so it cannot be a product of smaller factors.



Historical Note: Nicolaus Copernicus (1473–1543), the German astronomer who gave us a mathematically coherent heliocentric ("sun-centered") theory of our planetary system, used almost no algebraic symbolism in his writings. Many other German mathematicians of his time did, however. For instance, Michael Stifel's 1544 book, *Arithmetica integra*, extended Rudolff's symbols to higher powers of the unknown. Stifel also used capital letters for other unknowns, in combination with these power symbols, in a somewhat awkward way. We have not pursued that on this activity sheet because it adds relatively little to the big picture of the development of modern algebraic notation.

3. Nicolas Chuquet (pronounced "shoo-KAY") was a medical doctor who wrote a manuscript on "the science of numbers" in 1484. Because his manuscript was unpublished for a long time, it was not well known to European mathematicians. Its ground-breaking advance in notational simplicity was not recognized for another century, when more prominent people began to use variants of it. Like others of his time, Chuquet confined his attention to powers of a single unknown. Unlike the others, however, his exponential notation showed explicitly that *there was only one unknown*, its powers being derived from it. In this way, he broke with the tradition of using unrelated symbols for its different powers, opening the way for later notational advances that would include more than one unknown.

The word *montent* translates as “amounts to” or “comes to.” Chuquet also used other words or phrases to denote equality from time to time.

(a) $5x^4 - x^2 + 7 = 2x^3 + x - 1$
$x^4 + x^3 + x^2 + x + 1 = 2x^5 - 32$

(b) $1^5.\bar{p}.3^2.\bar{m}.1.$ *montent* $4^4.\bar{m}.2^1.\bar{p}.8$
$35^2.\bar{m}.7^1.\bar{p}.5.$ *montent* $1^3.\bar{m}.1^2.\bar{p}.2^1.\bar{p}.2.$

4. The word for “root” is *radix* in Latin and *racine* in French, so Chuquet used the symbol $\mathcal{R}$ for it, as was customary at the time. What was *not* customary, however, was his understanding that square and cube roots were not fundamentally different kinds of things, but rather that roots of all degrees are essentially the same kind of thing. In that respect, he anticipated common understanding by more than a century.

(a) $\sqrt{x^2 - 6x + 9} = x - \sqrt{9}$
$x^4 + 5x^3 - \sqrt{4x} = \sqrt[3]{x^2 + \sqrt{12x}}$

(b) $\mathcal{R}^2.\underline{1^2.\bar{p}.10^1.\bar{p}.\mathcal{R}^2.625}.$ *montent* $.1^1.\bar{p}.\mathcal{R}^2.25.$
$.7^4.\bar{m}.3^3.\bar{p}.\mathcal{R}^3.\underline{2^2.\bar{m}.9^1}.\bar{p}.\mathcal{R}^5.\underline{8^1.\bar{m}.1}$

(c) In modern notation:
$\sqrt{4x^2 + 24x + 36} - x - \sqrt{16}$
$\sqrt{4}\sqrt{x^2 + 6x + 9} - x - 4$
$2\sqrt{(x + 3)^2} - x - 4$
$2x + 6 - x - 4 = x + 2$, which is $.1^1.\bar{p}.2.$ in Chuquet’s notation.

5. Simon Stevin was a Flemish military engineer and inventor, a practical man who focused on using mathematics in practical settings. His 1585 book, *The Tenth*, popularized the use of decimals for fractions, arguing that writing fractions as decimals allows operations on fractions to be carried out using the much simpler algorithms of whole-number arithmetic. His notation for decimals is similar to his algebraic notation. He wrote $5/4 = 1.25$ as 1⓪2①5②, and the same circled numbers show up in his algebraic notation. (This suggests, in particular, that he never used decimals when doing algebra!) Stevin’s signs for + and − were a lot like ours, but his signs for multiplication and division were M and D.

(a) $3x^5 - 7x^3 + 9x^2 - 11x$

(b) 6④ + 1② — 1① + 5

(c) These exercises in exponent arithmetic are easier to understand in modern notation, as follows.
$3x^2 \times 3x^4 = 3x^6$ is FALSE. $3x^2 \times 3x^4 = 3x^8$ is FALSE.
$3x^2 \times 3x^4 = 9x^6$ is TRUE. $x^2 \times x^4 = x^6$ is TRUE.

Sheet 1-4: Letters for Numbers

• Main Feature •
Writing more than one unknown

A major breakthrough in notational flexibility was made by François Viète at the end of the 16th century. Viète was a lawyer, a mathematician, and an advisor to King Henri IV of France with duties that included deciphering messages written in secret codes. His mathematical writings focused on methods of solving algebraic equations. To clarify and generalize his work, he introduced a revolutionary notational device. In Viète's own words:

> In order that this work may be assisted by some art, let the given magnitudes be distinguished from the undetermined unknowns by a constant, everlasting and very clear symbol, as, for instance, by designating the unknown magnitude by means of the letter A or some other vowel... and the given magnitudes by means of the letters B, G, D or other consonants.[4]

Using letters for both constants and unknowns allowed Viète to write general forms of equations instead of relying on specific examples. Earlier mathematicians tended to write in terms of concrete examples, leaving readers to check that the methods explained would actually work in all cases. There was always the danger that a method might work in a special case but not in general. By working with letters, Viète could check all cases at once.

Some earlier writers had experimented with using letters, but Viète was the first to use them as an integral part of algebra. It may be that the emergence of this powerful device was delayed because the Hindu-Arabic numerals were not commonly used until well into the 16th century. Prior to that, Roman numerals (and Greek numerals before them) were used, and these systems used letters of the alphabet for specific quantities.

As soon as equations contained more than one unknown, it became clear that the old exponential notation was insufficient. It would not do to write $5^4 + 7^3$ if one meant $5A^4 + 7E^3$. In the 17th century, several competing notational devices appeared almost simultaneously. In the 1620s, Thomas Harriot in England would have written it as $5aaaa + 7eee$. In 1634, Pierre Hérigone of France wrote unknowns with coefficients before and exponents after, as in $5a4 + 7e3$. In 1636, James Hume published an edition of Viète's algebra with exponents elevated and in small Roman numerals, as in $5a^{iv} + 7e^{iii}$. In 1637, a similar notation appeared in René Descartes's

[4] From Viète's *In artem analyticam Isagoge* of 1591, as translated by J. Winfree Smith.

La Géométrie, but with the exponents written as small Hindu-Arabic numerals and x and y for the unknowns, as in $5x^4 + 7y^3$. Of these notations, Harriot's and Hérigone's were the easiest to typeset, but could be hard to read; what power is $aaaaaaaaaa$, after all? In the end, clarity won out over typographical convenience and Descartes's method eventually became the standard notation used today.

Solutions

1. The mathematical issue here is recognizing the difference between variables and constants.

   (a) $5x + 6 = 4y - 2$

   (b) Viète: *B in A* $+$ *C in E aequ. D.* Other consonants (not vowels) may be used in place of B, C, and D.
   Modern: $ax + by = c$

   (c) *E aequ. M in A* $+$ *B.* (Other consonants may be used for M and B, and other vowels may be used for E and A.)

   (d) There are infinitely many correct answers for this, of course. Here are two: $2x - 3y + 4 = 5$ and $7x - 5y + 12 = 163$. Negative constants should be avoided because they were not fully accepted as legitimate numbers in Viète's time, but fractions and irrational roots would be OK, too.

2. This notation is an intermediate stage between Viète's words and our modern exponential notation. Harriot used a low dot between the numerical coefficient and the letters for the unknown just as a typographical separator, as was the custom of his time, not as a symbol for multiplication.

   (Note: Harriot died in 1621 and his book was put together posthumously from his manuscript notes. The editor took many liberties with the notes, some of which he seems to have misunderstood. In particular, the notations in the book are different from the ones in Harriot's manuscripts. The reference in the activity sheet is to the book. In the manuscripts, the equals sign has two small vertical strokes connecting the two parallel lines, and the "less than" and "greater than" signs also have a vertical stroke. The printer (or editor?) probably decided to go with what was easily available, such as a sideways "V" for the inequality symbols.)

   (a) $5a^3 + 2a^2 - 4a = 9$. You can also use x in place of a, the usual first-choice symbol for an unknown in today's notation, which is a tradition begun by Descartes.

   (b) This is an exercise in recognizing the difference of squares. $4a^2 - 9 = (2a + 3)(2a - 3)$, so $a = \pm\frac{3}{2}$. Harriot probably would have discarded the negative answer as "false"; negative numbers were not fully accepted as legitimate numbers in the 17th century, particularly as roots.

(c) False. This is just a simple question to see if students understand powers or if they are guessing based on tempting visual patterns. $3a^2+6a^2 = 9a^2$, not $9a^4$.

3. These are a few common-sense exercises in reasoning about inequalities. You might mention the similarity between Harriot's inequality symbols and the sign used in musical notation for "loud to soft" and "soft to loud."

   (a) No. $a^2 < a^3$ for all $a > 1$. When $a < 1$ the inequality is reversed (because multiplying by a positive number less than 1 makes the product smaller). When $a = 1$, $a^2 = a^3$.

   (b) Since a is positive, The reduces to $a^2 > 7$, so $a > \sqrt{7}$.

   (c) Since aa is always positive, except for $a = 0$, this reduces to $0 < a^2 < 1$. That is, $-1 < a < 1$ and $a \neq 0$.

4. The work of French mathematicians in the 1630s, from Hérigone to Hume to Descartes, established the exponential notation we use today.

   (a) $7a^4 + 2a^3 - a^2 = 2a - 1$
   $a^5 - 3a^4 - a^3 + 7a = 5a^2e + ae^2 - 2e$

   (b) $a4 \sim 4a3 + 2a2 \sim 3a + 5\ est\ 0$
   $a3 + 3a2e + 3ae2 + e3\ est\ 2a2e2 \sim 1$

   (c) Translation into modern notation makes this obvious.
   $2a2e + 2ae2\ est\ 4a3e3$ is $2a^2e + 2ae^2 = 4a^3e^3$, not correct.
   $2a2e2 + a2e2\ est\ 3a2e2$ is $2a^2e^2 + a^2e^2 = 3a^2e^2$, correct.

5. In an earlier book, Hume had followed the custom of writing exponents of the unknown without the base. He used in-line lowercase Roman numerals for the exponents so that they could be typeset in-line without being confused with coefficients. For instance, he wrote $4x^3 + 2x^2$ as $4iii + 2ii$. Raising the exponents above the baseline made them visually distinct from the coefficient, making the formulas much easier to read.

   (a) $A^5 - 2A^4 + 3A^3 - 5A^2$

   (b) $A^{vi} + 3A^{iv} - 5A^{iii} + 2A^{ii}$

6. We have used powers higher than 3 to illustrate various notations in this activity sheet, but that usage was not common prior to the work of Descartes. For centuries, powers of an unknown quantity were confined by the geometric intuition of squares and cubes, and the notation reinforced that confinement. But Harriot saw that higher powers were just products when he wrote a^4 as

aaaa, for example. Building on the advances of his century, Descartes finally found a way to convince mathematicians that x^2 could just as well be the length of a line as the area of a square. Once that was clear, x^2 and x^3 could be treated as magnitudes independent of geometric dimension, thereby giving a new legitimacy to x^4, x^5, x^6, and so on.

(a) $axx + bx + c \ \propto\ y^3$ -- 1

(b) $2x^2 = 6x - 4$. Rewrite as $2x^2 - 6x + 4 = 0$, divide by 2, and factor into $(x-1)(x-2) = 0$, so $x = 1$ and $x = 2$ are the solutions.

7. (a) $aaa - 3.aae + 3.aee = eee$

(b) $a3 \sim 3a2e + 3ae2$ *est* $e3$

(c) $A^{iii} - 3A^{ii}E + 3AE^{ii}$ *donne* E^{iii}

(d) $x^3 - 3xxy + 3xyy \ \propto\ y^3$

2

Linear Thinking

Mathematical Focus	**Ratio, Proportion, and Slope**
Historical Connections	Egypt, 1650 BCE China, 100 BCE Europe, 13th century England and America, 1600–1900

Linearity, the preservation of ratio, is one of the truly foundational ideas in all of mathematics. It is a unifying thread running from proportionality in arithmetic to first-degree equations and their pictures in the coordinate plane, and then to the basic trigonometric functions, differential calculus, and beyond. Recognition of its importance in the everyday affairs of commerce and craftsmanship preceded the development of symbolic algebra by many centuries. It is not surprising, then, that a variety of calculational devices for its application emerged as basic tools of arithmetic. Two of the most widely known such devices are the *Rule of Three* and *False Position.* They are the focus of this chapter.

So why teach ways to sidestep the usual methods for solving linear equations? Two reasons: (1) Using these unfamiliar methods requires students to pay careful attention to the relevant details of a problem, and (2) analyzing why they work provides a deeper understanding of what a linear equation says. In particular, the Rule of Three forces students to recognize the kinds of things that numbers in a problem or application represent, and explaining why False Position works can lead to a deeper understanding of slope and the determination of a linear function from two of its points. Both rules illustrate the effectiveness of proportional reasoning.

These activities presume that students have already studied the basic facts about linear equations. They are intended to deepen their understanding of those facts.

Sheet 2-1: The Rule of Three Direct

• Main Feature •
Direct proportion

The mathematical principle of proportionality arises from a common-sense intuition. Most of us, even as young children, have an intuitive idea of "fair" distribution of something — sharing candy, being paid for work, inheriting money, etc. The algebraic definition of proportion is just a way to make this idea more precise.

Even without algebra, proportionality has been one of the key principles of commercial arithmetic for many centuries. Merchants, traders, and bankers have used it for pricing goods, exchanging currency, setting wages, and many other things. This widespread utility prompted the development of a rule for calculating proportional values "automatically" in a variety of circumstances. The algorithm became known as the **Rule of Three**.[5] It was a standard topic that appeared virtually everywhere arithmetic was taught — in Chinese writings of 100 BCE, in India of the 5th century, in the 8th-century Arabic writing of al-Khwarizmi, in Europe of the early Renaissance, and in arithmetic schoolbooks of England and America, right into the 20th century. In all these different times and cultures, the rule has been stated in essentially the same way. For instance:

> China, c. 100 BCE, *The Nine Chapters on the Mathematical Art*:
> "Take the given number to multiply the sought rate. The product is the dividend. The given rate is the divisor. Divide." ([12] p. 141)
>
> India, 1150, *Lilivati* by Bhaskara II:
> Calling the three given terms the *argument*, *fruit*, and *requisition* in that order, and the unknown number the *demand*, it says: "The first and last terms. . . must be of like denomination, the fruit which is of a different species, stands between them. And that, being multiplied by the requisition and divided by the first term, gives the fruit of the demand." ([12] p. 136)
>
> Italy, 1202, Fibonacci's *Liber Abbaci*:[6]
> "[F]our proportional numbers are always found in all negotiations of which three are known and one is left truly unknown; the first indeed of these three unknown numbers is the number of the sale of any merchandise,. . . or weight, or measure. . . . The second moreover is the price of the sale that is the first

[5] There is a Rule of Three in many other fields, too, unrelated to this one. A quick Internet query will turn one up in areas as diverse as speechwriting and witchcraft.

[6] This is also called *Liber Abaci*, particularly in the title to the reference cited. There is disagreement among scholars as to which is more accurate.

number.... Often a third will be some of the same sale of a quantity of merchandise for which the price, namely the fourth number, is unknown .... [If] the multiplication of the second quantity by the third you divide by the first, then certainly the fourth quantity results from the division...." ([13] p. 128)

USA, 1809, Nicolas Pike's *A New and Complete System of Arithmetick*:
"THE Rule of Three Direct: teacheth, by having three numbers given, to find a fourth that shall have the same proportion to the third, as the second hath to the first.
Rule. 1. State the question by making that number, which asks the question, the third term;...that, which is of the same name or quality as the demand, the first term; and that which is of the same name or quality with the answer required the second term.
2. Multiply the second and third terms together, divide the product by the first, and the quotient will be the answer to the question...." ([10] pp. 43–44)

The Rule of Three quotes on this activity sheet and the next are from Nathan Daboll's *Schoolmaster's Assistant*, the most widely used arithmetic schoolbook in the United States during the first half of the 19th century.[7] It was first published in 1799 and republished in various editions well into the 1840s. The copy from which these quotes are taken was printed in 1843; it is [6] in the bibliography.

Solutions

1. This is an everyday example to illustrate the idea of proportional relationship.
   (a) \$32 (b) \$24 (c) \$22
   (d) Look for some sense of an hourly rate here.

Important: Please do NOT have your students write a proportion as equal fractions at this time. There are two reasons for this, one conceptual and one algorithmic:

- Since a ratio often involves quantities of unlike kinds of things, its expression as a fraction invites confusion. For instance, 5 apples for \$2 is a ratio 5:2, but the quantity $\frac{5}{2}$ is neither apples nor dollars; it's an abstract rate, apples per dollar.
- The older four-numbers-in-line notation is essential to the form of the Rule of Three as it has been stated for many centuries.

2. (a) (1a) $2 : 16 :: 4 : 32$ (1b) $2 : 16 :: 3 : 24$ (1c) $2 : 16 :: 2\frac{3}{4} : 22$
   It's OK, but not necessary, for students to include the denomination labels (hours, dollars) in these proportions. The last one requires a little care to be sure that the same time unit is being used in both ratios.

[7] It was so well known that Herman Melville mentioned it (as "Daboll's arithmetic") in his 1851 novel, *Moby Dick*.

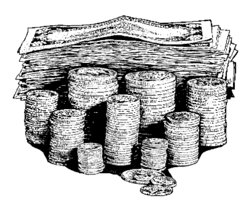
(b) Many different examples are appropriate here. Some examples: 4 hours is to 240 miles as 6 hours is to 360 miles; 4 hours is to 240 minutes as 6 hours is to 360 minutes; 4 jackets is to \$240 as 6 jackets is to \$360; etc.

(c) The order of the numbers in the proportion statement is the real issue behind this very easy calculation. The cost is \$9, so the proportion is $5 : 3 :: 15 : 9$; that is, 5 lbs. : \$3 :: 15 lbs. : \$9.

3. **(1b)** The three numbers, in order, are 2, 16, and 3. By the Rule of Three, $(16 \times 3) \div 2 = 48 \div 2 = 24$ (dollars).

   (1c) The issue here is that the first and third numbers must be "of the same kind." Either the 45 min. needs to be converted to $\frac{3}{4}$ hr. or the hours should be converted to minutes. Using the first case, the numbers are 2, 16, and $2\frac{3}{4}$, so $(16 \times 2\frac{3}{4}) \div 2 = 44 \div 2 = 22$ (dollars).

About Colonial Currency: Commerce in colonial America was complicated by the lack of a uniform system of money. Colonial merchants had to deal with currency from many different countries. The problems in this sheet focus on the two most common kinds, the Spanish dollar and the pound. Minted in Mexico and Peru from Spanish American silver, the Spanish dollar was worth eight *reales*, or *bits.* (A *real* [pronounced "ray-AL"] was an old Spanish coin.) It was often called a *piece of eight*[8] and was sometimes broken into 1-, 2-, or 4-bit pieces.[9] The system brought from England by the colonists was built around *pounds* (£), *shillings* (*s*), and *pence* (*d*), based on the Roman *librae*, *solidi*, and *denarii.*[10] There were 20 shillings in a pound, and 12 pence in a shilling. However, each colony had the right to regulate its own currency, so "the" pound had different values in different places! Even after the American Revolution, the Articles of Confederation preserved that right for the individual states. It was well into the 19th century before the U.S. dollar (\$) became the uniform standard for money throughout the United States. (This subject can be the basis of some rich interdisciplinary projects.)



Wrestling with the annoyances of counting by dozens (12s) and scores (20s) in the following activities will give your students several benefits:

- practice with arithmetic skills and number sense in real historical contexts;
- a deeper appreciation of the power of our decimal system of currency; and
- a better understanding of money references in history and literature.

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[8]Pieces of eight appear in *Treasure Island*, *Pirates of the Caribbean*, and other pirate stories.

[9]This is the origin of "two bits" as the modern nickname for a quarter of a dollar.

[10]The *libra* was a measure of weight. The English denomination symbols £, *s*, and *d* come from these Latin names, as does our abbreviation *lb.* for pound.

4. These are problems 47 and 44, respectively, on page 50 of [10]. We have quoted Pike faithfully, despite the more common practice of putting the pound sign before the number.

   (a) The three numbers are 320, 400, 1 (pounds). $(400 \times 1) \div 320 = 1\frac{1}{4}$, which is $1£\,5s$.

   (b) The three numbers in proper order are 125, 100, 437. $(100 \times 437) \div 125 = 349.6$, which is $349£\,12s$.

5. The data source for this activity is [9]. Perhaps the easiest way to do these is to convert everything to pence. (The early arithmetic books recommend that strategy.)

   British: The three quantities are $7s\,6d, 4s\,6d, £1$. Converting to pence and applying the Rule of Three yields $(54 \times 240) \div 90 = 144$ pence, or 12 shillings.

   Massachusetts: $7s\,6d, 6s, £1$ yields $(72 \times 240) \div 90 = 192d = 16s$

   New York: $7s\,6d, 8s, £1$ yields $(96 \times 240) \div 90 = 256d = £1\,1s\,4d$

6. Choosing the three numbers in the proper order requires a little thought. Convert the bill to pence and the dollars to bits. Then the three quantities are 58 bits, 522 pence, 8 bits, so $(522 \times 8) \div 58 = 72d = 6s$. This is a historically accurate answer.

7. This activity connects direct proportions with linear equations.

   (a) By the Rule of Three Direct, $\frac{x}{15} = \frac{3}{5}$, so $x = 15 \times \frac{3}{5}$, so $x = \frac{15 \times 3}{5} = 9$

   (b) The statement of what the Rule "teaches" dictates this equality of ratios.
   $$\frac{x}{n_3} = \frac{n_2}{n_1}$$

   (c) This generalizes part (a).
   $$\frac{x}{n_3} = \frac{n_2}{n_1}, \quad \text{so } x = \frac{n_2}{n_1} \times n_3, \quad \text{so } x = \frac{n_2 \times n_3}{n_1}$$

Sheet 2-2: The Rule of Three Inverse

• MAIN FEATURE •
Inverse proportion

This sheet presumes that students are familiar with the Rule of Three Direct from sheet 2-1. The most troublesome part of using the Rule of Three Inverse is the setup step. The three numbers must be put in their proper order; the two quantities representing the same kind of thing must be in the first and third positions.

Solutions

1. The point here is *not* to solve the problem, but to see that the Rule of Three Direct is an inappropriate tool for it. Student explanations may vary, but should contain some sense that the proportionality from the Rule of Three Direct would result in a larger number (a longer time), which doesn't make sense. A faster speed should result in a shorter time. Explicitly, if this problem is set up as the Rule of Three Direct says, the three numbers are 54 (mph), 4 (hours), and 60 (mph). The computation is $(60 \times 4) \div 54 = 4\frac{4}{9}$ (hours), a longer time for a faster speed!

The Rule of Three Inverse description on this sheet is quoted from [6], pages 97–98, with boldface emphasis added. It appears in virtually the same form on page 76 of [15] (who calls it "Inverse Proportion") and on pages 52–53 of [10].

2. (a) Students may need to be reminded of the proper setup from the Rule of Three Direct: "Place the numbers so that the first and third terms may be of the same kind; and the second term of the same kind with the answer, or thing sought." In this case, the numbers in proper order are 54 (mph), 4 (hours), and 60 (mph). Then $(54 \times 4) \div 60 = 3.6$ hours.

   (b) Help students to see that this is a way for them to check their work. If the answer is correct, both computations will yield the same distance: $54 \times 4 = 216$ mi., and $60 \times 3.6 = 216$ mi.

3. These problems are quoted directly from pages 98–99 of [6].

   (a) The three quantities in proper order are 20 days, 12 men, 8 days, so $(20 \times 12) \div 8 = 30$ men.

   (b) The three quantities in proper order are 50 cents, 30 bushels, 75 cents, so $(50 \times 30) \div 75 = 20$ bushels.

   (c) This question asks students to take one additional step, which is easy to overlook. The three quantities in proper order are 2 mo., 650 men, 5 mo., so $(2 \times 650) \div 5 = 260$ men *who can remain.* Therefore, $650 - 260 = 390$ men must leave.

4. This question is a twin to #7 of sheet 2-1. It connects inverse proportions with linear equations. The algebraic formulation highlights a troublesome asymmetry between the Rule of Three Direct and the Rule of Three Inverse as they were traditionally taught.

(a) By the Rule of Three Inverse, $\frac{x}{12} = \frac{20}{8}$, so $x = \frac{20}{8} \times 12 = 30$.

(b) The statement of what the Rule "teaches" dictates this equality of ratios.
$\dfrac{x}{n_2} = \dfrac{n_1}{n_3}$

(c) This generalizes part (a).
$\dfrac{x}{n_2} = \dfrac{n_1}{n_3}$, so $x = \dfrac{n_1}{n_3} \times n_2$, so $x = \dfrac{n_1 \times n_2}{n_3}$

5. This question is about pattern recognition. It may be a bit too abstract for some students, but it is a good opportunity for some habit-of-thought training. The two equal-fractions forms, dictated by the wording of the rules, are:

$$\text{Direct: } \frac{x}{n_3} = \frac{n_2}{n_1} \;\; ; \;\; \text{Inverse: } \frac{x}{n_2} = \frac{n_1}{n_3}$$

The cross-multiplication test for equal fractions allows us to rewrite the Direct equation by interchanging n_2 and n_3 to get:

$$\text{Direct: } \frac{x}{n_2} = \frac{n_3}{n_1} \;\; ; \;\; \text{Inverse: } \frac{x}{n_2} = \frac{n_1}{n_3}$$

Now the left sides of both equations are the same, but the right sides are inverses of each other!

6. All of these problems are quoted from [6], pages 91–98. Students are being asked only to label each one as *direct* or *inverse*. Nevertheless, in case you want them to actually solve any of these problems by the Rule of Three, the solutions are included here. In each solution, the three numbers are listed first, in their proper order. (Notice that some of these problems contain numbers that are not involved in the computation!)

(a) Inverse. 12 months, 100£, 8 months. $(12 \times 100) \div 8 = 150£$

(b) Direct. Convert to shillings and months. 12 months, 1510 shillings, 1 month. $(1510 \times 1) \div 12 = 125\frac{10}{12}s = 6£\,5s\,10d$

(c) Direct. 7 yds., 1547 cents, 12 yds. $(1547 \times 12) \div 7 = \$26.52$

(d) Inverse. 12 hours, 5 months, 8 hours. $(12 \times 5) \div 8 = 7\frac{1}{2}$ months.

(e) Direct. (Note: A *tun* is actually a measure of liquid. Daboll may have meant "ton.") 3 cattle, $4\frac{1}{2}$ tuns, 25 cattle. $(4\frac{1}{2} \times 25) \div 3 = 37\frac{1}{2}$ tuns

(f) Inverse. 12 hours, 5 days, 10 hours. $(12 \times 5) \div 10 = 6$ days.

(g) Direct. Convert to shillings (or pence, if you prefer). $12\frac{1}{2}$ shillings, 1 week, 650 shillings. $(1 \times 650) \div 12\frac{1}{2} = 52$ weeks $= 1$ year.

Note: According to the dictionary, *ratio* and *rate* come from different etymological roots. The usual dictionary definition of ratio is inaccurate, in that it restricts the idea to a relationship between numbers "of the same kind." However, when it is used with denominate numbers, as in commercial transactions or in science, the kinds often are very different (see above). Even ancient Greek mathematics reflects the more general usage. In Euclid, for example, ratios start off as between magnitudes of the same kind, as in the areas of two triangles with the same height being in the same ratio as their bases. But Euclid soon proves that $a : b = c : d$ implies that $a : c = b : d$, at which point the temptation to ignore the "same kind" restriction becomes irresistible.

Sheet 2-3: False Position

• MAIN FEATURE •
Solving proportionality problems

Problems that reduce to solving an equation of degree one arise naturally whenever we apply mathematics to the real world. It's not surprising, then, to find that almost everyone who studied mathematics, from the Egyptian scribes to the Chinese civil servants, developed techniques for solving such problems. Many, many centuries before the emergence of symbolic algebra and coordinate geometry, these problems were solved by a method known as *false position*. "False position" does not mean "wrong place"; it means "wrong guess." Here the word *position* comes from *posit*, which means to assert or state something as a fact or hypothesis. The method depends on making a wrong guess and then using it to find the right answer. This is "trial and error" at its productive best. The initial try expects an error and uses it to find the correct answer. Maybe it is time bring this expression back into favor, instead of the dead-end phrase, "guess and check." Students should be encouraged to view errors as information to be used, not just mistakes to be avoided.

As students work through these activities, they will be challenged to see how the old methods connect with what they are learning about solving linear equations algebraically and representing them graphically. In particular, their work should lead to a deeper understanding of how the equation for a straight line can be derived from any two points on it. It might also give them a better appreciation of the efficiency of algebra!

Solutions

1. This problem comes from the Rhind Papyrus of about 1650 BCE, also known as the Ahmes Papyrus. It is a collection of problems probably used for training

young scribes in Ancient Egypt. Some of its problems are quite a bit more complicated that these two. This equation is $x + \frac{1}{4}x = 15$, or $\frac{5}{4}x = 15$, so $x = \frac{4}{5} \times 15 = 12$.

2. (a) Guess at something that is easy to take a half and a third of; say 6. Then $6 + 3 + 2 = 11$. We want 44, which is 4 times 11, so the correct answer is 4 times our guess, which is 24.

   (b) $x + \frac{1}{2}x + \frac{1}{3}x = 44$, so $\frac{11}{6}x = 44$, so $x = \frac{6}{11} \times 44 = 24$.

3. These problems are taken directly from pages 125–126 of [15], adjusted slightly to remove ambiguity. Students should verify their answers by substituting them back into the problems to see if the conditions are satisfied.

   (a) Guess at something that is easy to divide into halves, thirds, and fourths — say 24, for example. (Many other numbers would work equally well.) Then $12 + 8 + 6 = 26$, so $26 : 24 :: 130 : answer$. By the Rule of Three, the correct answer is $(24 \times 130) \div 26 = 120$ dollars.
   Verify: $\frac{1}{2}120 + \frac{1}{3}120 + \frac{1}{4}120 = 60 + 40 + 30 = 130$.

   (b) Try 40. Then $40 + 40 + 20 + 10 = 110$, so $110 : 40 :: 264 : answer$. By the Rule of Three, $(40 \times 264) \div 110 = 96$ students.
   Verify: $96 + 96 + 48 + 24 = 264$

   (c) Try something easily divisible into tenths for the age of A; say 20. Then $20+30+2\frac{1}{10}\times(20+30) = 155$. $155 : 20 :: 93 : answer$, so $(20\times 93)\div 155 = 12$. That is, A is 12 years old, so B is 18, and C is 63.
   Verify: $12 + 18 + 2\frac{1}{10} \times (12 + 18) = 93$

Historical Note: Here are some facts from American history to connect the time that the arithmetic book cited in #3 was being used with some then-current events. If you wish, they can take the form of questions that students might answer via a quick Internet query.

Q: *In 1847, who was President of the United States?*
A. James K. Polk was in office as the 11th President.

Q: *What were some of his achievements?*
A. He engineered the peaceful annexation of the Northwest Territory (now including the states of Washington and Oregon) by compromising on the 49th Parallel as the border with Canada, thereby avoiding the "Fifty-four forty or fight" confrontation urged by some extreme expansionists. He also precipitated a brief war with Mexico, resulting in the acquisition of the territory that is now California, Nevada, Utah, most of Arizona, and parts of New Mexico, Colorado, and Wyoming. Some students may cite other facts about President Polk.

Q: *How many states were there then?*
A. In 1847 there were 29 states.

Q: *Which state joined the Union next, and when?*
A. Wisconsin became the 30th state in 1848.

Q: *Was this before or after the Civil War?*
A. Before. The Civil War lasted from 1861 to 1865.

4. This question gives students some practice translating "word problems" into algebraic form. It also illustrates how algebraic notation can simplify and clarify situations.

   (a) $\frac{1}{2}x + \frac{1}{3}x + \frac{1}{4}x = 130;\ \ \frac{13}{12}x = 130;\ \ x = \frac{12}{13} \cdot 130 = 120$

   (b) $2x + \frac{1}{2}x + \frac{1}{4}x = 264;\ \ \frac{11}{4}x = 264;\ \ x = \frac{4}{11} \cdot 264 = 96$

   (c) The unknown in this equation represents the age of A.
   $x + 1.5x + 2.1(x + 1.5x) = 93;\ \ 7.75x = 93;\ \ x = 93 \div 7.75 = 12$

5. This question paves the way for undertanding double false position, the topic of the next activity sheet.

   (a) $y = \frac{13}{12}x$ (b) $y = \frac{11}{4}x$ (c) $y = 7.75x$

   (d) In all three cases, the y-intercept is $(0, 0)$. This links with #6, which exposes the limitations of the (single) false position method.

   (e) (a) $\frac{13}{12}$ (b) $\frac{11}{4}$ (c) 7.75

   (f) Students may express this idea in a variety of ways. In each case, the slope is the ratio of the second term to the first, which is also the ratio of the answer to the third term.

6. (a) If we use \$50 for A's "certain sum," then $50 + 60 + (50 + 60) = 220$, so $220 : 50 :: 500 : \mathit{answer}$. By the Rule of Three, $(50 \times 500) \div 220 = 113\frac{7}{11}$ dollars.
   Verify: $113\frac{7}{11} + 123\frac{7}{11} + (113\frac{7}{11} + 123\frac{7}{11}) = 474\frac{6}{11}$ dollars, *not* 500 dollars. It didn't work!

   (b) Let x represent A's contribution. Then $x + (x + 10) + (2x + 10) = 500$; so $4x + 20 = 500,\ \ x = \frac{500-20}{4} = 120$ dollars.

   (c) $y = 4x + 20$; slope is 4; y-intercept is $(0, 20)$

   (d) The y-intercept is not $(0, 0)$; that is, this equation has a nonzero constant term. (The significance of this will become clearer in Sheet 2-4.)

Sheet 2-4: Double False Position

• Main Feature •
Solving general linear problems

The false position method described on sheet 2-3 can be applied only to equations of the form $Ax = B$. If, instead, the equation is of the form $Ax + C = B$, then it is no longer true that multiplying x by a factor causes B to change by the same factor, and the process breaks down. We might try subtracting C from both sides, but that isn't always easy, because the expression on the left side might initially be very complicated, and finding the correct constant to subtract would require us to simplify it to the form $Ax + C$. Moreover, in the days before symbolic algebra, expressing such problems as linear equations was not even an option.

Instead, a way was found to extend the basic idea to equations of that type without any such algebraic manipulation. It is called **double false position**. This is such an effective method for solving linear equations that it continued to be used long after the invention of algebraic notations. Since it doesn't require any algebra, it was taught as an arithmetic skill well into the 19th century.

As its name implies, double false position is based on making *two* wrong guesses. The key idea behind it is that a straight line has a constant slope. Before people had coordinate geometry in hand, this was expressed by saying that the *difference* between two values of $Ax+C$ was proportional to the *difference* between the values of x. If $y = Ax + C$ is the underlying equation, that just says that the difference in y-coordinates is proportional to the difference in x-coordinates. The constant of proportionality is exactly A, the slope of the line, but double false position never actually computes A. Instead, it uses the fact that two ratios are equal. (The different instructions for "alike" and "unlike" errors in the Daboll quotation are just a way to avoid dealing with negative numbers.)

Solutions

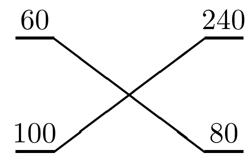
Students will need scrap paper for their calculations in some of these questions.

1. This question creates the data needed to carry out the double false position process in #2.

   (a) $60 + 70 + (60 + 70) = \$260$. It is too small by $500 - 260 = \$240$.

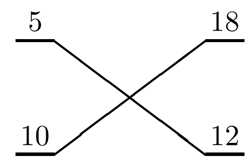
   (b) $100 + 110 + (100 + 110) = \$420$. It is too small by $500 - 420 = \$80$.

2. (Among other things, this is an exercise in reading and following instructions carefully!) In this case, the errors are alike, so the calculation is



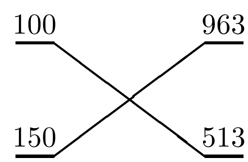
$((100 \cdot 240) - (60 \cdot 80)) \div (240 - 80)$
$(24,000 - 4800) \div 160 = 19,200 \div 160 = 120$
So A paid \$120, B paid \$130, and C paid \$250.
Check: $\$120 + \$130 + \$250 = \$500$, as required.

3. Students' solutions will depend on their initial guesses, of course. Here is a typical example.



Guess 1: 5. Then $(5 + 10 - 5) \times 2 = 20$; but there should be 38, so the error is 18 (too few).
Guess 2: 10. Then $(10 + 20 - 5) \times 2 = 50$; but there should be 38, so the error is 12 (too many).
Cross-multiply: $5 \times 12 = 60$ and $10 \times 18 = 180$. Since the errors are unlike (the first too few, the second too many), the answer is $(60 + 180) \div (18 + 12) = 8$ people on the bus when it started. Check: $(8 + 16 - 5) \times 2 = 38$, as required.

4. Again, students' solutions will depend on their initial guesses. Here is a typical example.



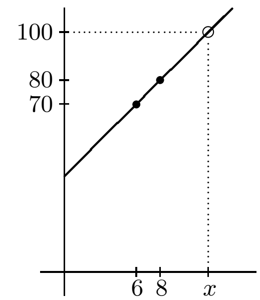
Guess 1: \$100. Then $((300 + 50) \times 3) - 10 = 1040$, which is too low by \$963.
Guess 2: \$150. Then $((450 + 50) \times 3) - 10 = 1490$, which is too low by \$513.
Cross-multiply: $100 \times 513 = 51,300$ and $150 \times 963 = 144,450$. Since the errors are alike (both too low), the answer is $(144,450 - 51,300) \div (963 - 513) = \$207$.
Check: $((3 \cdot 207 + 50) \times 3) - 10 = \$2003$, as required.

To see why double false position works, students need to re-examine some fundamental concepts:
— slope as a constant of proportionality
— a first-degree equation as a straight-line function
— the graph of a linear function
— the two-point determination of a linear function
This gives you a novel context for unifying and strengthening their understanding of these important algebraic ideas.

5. (a) $x + (x + 4) + (x + 12) + 2(x + 12) = 100$. Its degree is 1. (You might want to ask students how they know.)

   (b) For guess 6: $6 + 10 + 18 + 2 \cdot 18 = 70$, so the error is 30.
   For guess 8: $8 + 12 + 20 + 2 \cdot 20 = 80$, so the error is 20.
   Cross-multiply: $6 \times 20 = 120$ and $8 \times 30 = 240$. Since the errors are alike (both too low), the answer is $(240 - 120) \div (30 - 20) = 12$ (dollars).

(c) See the display at right.



(d) $\dfrac{100-70}{x-6} = \dfrac{100-80}{x-8}$, or $\dfrac{30}{x-6} = \dfrac{20}{x-8}$.

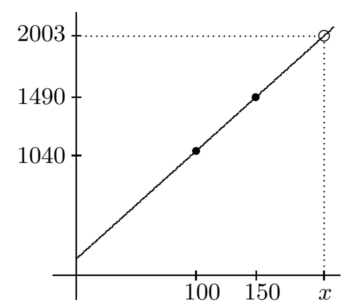
(e) $30(x-8) = 20(x-6)$

(f) Simplify the previous step:
$(30-20)x = (30 \times 8) - (20 \times 6)$, so
$x = \dfrac{(30 \times 8) - (20 \times 6)}{30-20} = \dfrac{120}{10} = 12$

(g) Help students notice that this final calculation is exactly the same as what they did in part (b)!

6. We presume that students have already solved #4 by double false position, so they have two guesses to work with. This solution uses the guesses from the sample solution to #4, above.

(c) In our solution, the two guess points are $(100, 1040)$ and $(150, 1490)$. Students will have to choose scales for the axes of their graph that will include their guess points and the answer point, as in this graph.



(d) $\dfrac{2003-1040}{x-100} = \dfrac{2003-1490}{x-150}$

$\dfrac{963}{x-100} = \dfrac{513}{x-150}$

(e) $963(x-150) = 513(x-100)$

(f) Simplify the previous step: $(963-513)x = (963 \times 150) - (513 \times 100)$, so
$x = \dfrac{(963 \times 150) - (513 \times 100)}{963-513} = \dfrac{93,150}{450} = 207$

(g) Again, this calculation is exactly the same as the double false position calculation.

3

A Square and Things

Mathematical Focus	Quadratic Equations
Historical Connections	Mesopotamia, 2nd millennium BCE Greece, 3rd century CE Islamic Empire, 9th century Europe, 16th–18th centuries

High school textbooks often teach students about solving quadratic equations like this:

- Write the equation with 0 on one side.
- Factor the (carefully chosen) quadratic into two linear factors by finding (whole) numbers that fit the coefficients.
- Set each linear factor to zero to find the two roots.

The Quadratic Formula comes along later, and completing the square is introduced (sometimes without pictures!) merely as a way to derive the formula. That approach is historically backwards. In fact, to factor the quadratic that way, you have to find two numbers from knowing their sum and their product (i.e., from the coefficient of x and the constant term). What happened historically was the opposite: given the sum and the product, mathematicians set up the quadratic equation and solved it by methods that — conceptually, at least — amounted to the Quadratic Formula.

Solving quadratic problems was an important part of mathematical lore many centuries before polynomial manipulation was even considered. The essential mathematical tools for writing and factoring polynomials did not exist until the 17th century, but mathematicians were solving problems involving quadratics more than two thousand years before that.

In earlier times, problems that we would solve with quadratic equations arose from geometric considerations, often as questions about area. The Mesopotamian scribes of the Old Babylonian period (1900–1600 BCE) became adept at solving such problems, some of which were quite artificial and seem to have existed solely to enable the scribes to exhibit their mathematical prowess. Their geometric approach foreshadowed that of the Greeks and probably directly influenced the mathematics of the Islamic Empire more than two millennia later, which, in turn, helped to shape European mathematics of the late Middle Ages.

This chapter examines some different ways in which people dealt with quadratic problems before the development of symbolic algebra, and it connects those ways with our current algebraic methods. These activity sheets presume that students have already learned what a quadratic equation is and know about the Quadratic Formula. The activities are intended to reinforce and deepen their understanding of these ideas.

Sheet 3-1: Completing a Square

• MAIN FEATURE •
The geometry of quadratics

The word "algebra" comes from *al-jabr w'al-muqābalah*, the title of a book written in Arabic around the year 825. The author, Muḥammad ibn Mūsa al-Khwārizmī, was probably born in what is now Uzbekistan. However, he lived in Baghdad, now the capital of Iraq but then the center of the Islamic empire. There the Caliphs encouraged learning and scholarship, establishing a kind of academy of arts and sciences called "The House of Wisdom." Al-Khwārizmī was a generalist; he wrote about geography, astronomy, and mathematics. But his book on algebra is one of his most famous. It starts out with a discussion of quadratic equations. In fact, he considers a specific problem:

> One square, and ten roots of the same, are equal to thirty-nine dirhems.[11] That is to say, what must be the square which, when increased by ten of its own roots, amounts to thirty-nine?

After al-Khwārizmī's time many other mathematicians wrote about quadratic equations. Their methods and their geometric justifications became more and more

[11] A *dirhem* was a unit of weight and a particular coin value.

sophisticated, but the basic idea never changed. In fact, even the example stayed the same. From the 9th century to the 16th century, books on algebra almost always started their discussion of quadratic equations by considering "a square and ten roots are equal to 39."

As your students begin to work through this sheet, you might point out to them that its title, *Completing a Square*, implies that there is only one square at issue. Algebraically, this means that the focus is on quadratics with leading coefficient 1.

Solutions

1. $x^2 + 10x = 39$

2. This should be an easy exercise in combinatorics, as well as a reminder of the customary distinction between variables and literal constants. In this context, "roots" and "things" are synonyms.

In words:	General form:
squares equal things	$ax^2 = bx$
squares equal a number	$ax^2 = c$
things equal a number	$bx = c$
squares and things equal a number	$ax^2 + bx = c$
squares equal things and a number	$ax^2 = bx + c$
squares and a number equal things	$ax^2 + c = bx$

   For the specific examples, any positive rational numbers can be used in place of a, b, and c, including an implicit 1 (e.g., $x^2 = 5$).
   Although it is one case of the modern generic form ($a = 0$), $bx = c$ is not really a quadratic.
   The problem in # 1 is of the form $ax^2 + bx = c$.

3. You halve the number of the roots, which in the present instance yields **five**. This you multiply by itself; the product is **twenty-five**. Add this to thirty-nine; the sum is **sixty-four**. Now take the root of this, which is **eight**, and subtract from it half the number of the roots, which is **five**; the remainder is **three**. This is the root of the square which you sought for; the square itself is **nine**.

4. Students will probably need some scrap paper for this. For your reference, we repeat the recipe here, with the new values plugged in:
   You halve the number of the roots, which in the present instance yields **3**. This you multiply by itself; the product is **9**. Add this to **40**; the sum is **49**. Now take the root of this, which is **7**, and subtract from it half the number of the roots, which is **3**; the remainder is **4**. This is the root of the square which you sought for; the square itself is **16**. Check: $4^2 + 6 \cdot 4 = 16 + 24 = 40$.

Note: If your students do not yet know the Quadratic Formula, they can skip #5 and #7 without affecting their ability to do #8.

5. The symbols a, b, and c are not used here in order to avoid confusion when the Quadratic Formula is derived on sheet 3-4. Instead, n represents "number of roots" and A reflects the fact that al-Khwārizmī thought of this as a measure of area. To apply the Quadratic Formula to this form, you need to think of it as $1x^2 + bx + (-c) = 0$. Then

$$x = \frac{-n \pm \sqrt{n^2 - 4 \cdot 1 \cdot (-A)}}{2 \cdot 1} = \frac{-n \pm \sqrt{n^2 + 4A}}{2}$$

6. You halve the number of the roots, which in the present instance yields $\frac{n}{2}$. This you multiply by itself; the product is $\frac{n^2}{4}$. Add this to A; the sum is $\frac{n^2}{4} + A$. Now take the root of this, which is $\sqrt{\frac{n^2}{4} + A}$, and subtract from it half the number of the roots, which is $\frac{n}{2}$; the remainder is $\sqrt{\frac{n^2}{4} + A} - \frac{n}{2}$. This is the root of the square which you sought for.

7. People of al-Khwārizmī's time did not consider negative square roots because their geometric interpretation (as the side length of a square) seemed absurd. Students should be able to convert their result in #6 into the formula in #5 by a step-by-step algebraic argument, more or less as follows.

$$\sqrt{\frac{n^2}{4} + A} - \frac{n}{2} = \sqrt{\frac{n^2 + 4A}{4}} - \frac{n}{2} = \frac{\sqrt{n^2 + 4A}}{2} - \frac{n}{2} = \frac{-n + \sqrt{n^2 + 4A}}{2}$$

8. The diagrams here are not drawn to scale. That is deliberate; their form is a generic representation of the "completing the square" argument. This reflects the spirit of al-Khwārizmī's work; he expected his examples to typify the procedure to be used with all problems of the same type. (You might want to mention to your students that the boxes inside the various regions represent areas; the outside boxes represent lengths.)

   Ancient mathematicians usually did not worry about drawing to scale or even drawing precisely. The diagram was meant to be a qualitative guide.

   (a) The bottom length is x. The area of the rectangle is $10x$.

   (b) Each bottom length is 5. Each rectangular area is $5x$.

   (c) The missing length is 5. As before, the rectangular areas are $5x$.

   (d) The area of the missing 5-by-5 square is $5^2 = 25$.

   (e) The area of the entire region without the missing corner is 39; it is the same as the area of the region we started with. The area of the completed square, then, is $39 + 25 = 64$. Its side length is $\sqrt{64} = 8$, so $x = 8 - 5 = 3$.

Sheet 3-2: Algebra Comes of Age

• Main Feature •
Zero and negatives in polynomials

The true power of completing the square did not emerge until algebraic notation and numerical understanding caught up with al-Khwārizmī's method. As noted in Chapter 1, the symbols of algebra were neither widespread nor uniform until well into the 17th century. Moreover, despite the tremendous post-Renaissance progress in science, commerce, and other uses of mathematics, there was continued resistance to negative numbers. In the 16th century, even such prominent mathematicians as Cardano in Italy, Viète in France, and Stifel in Germany rejected negative numbers as "fictitious" or "absurd." In the late 16th century that tide finally began to turn, as the usefulness of negative numbers became too obvious to ignore. (For instance, zero and negative coefficients allow Cardano's thirteen different types of cubic equations to be reduced to just one general type!)

This broader, more abstract understanding of the nature of numbers also included the legitimacy of zero as a number, despite the fact that it did not represent a "quantity" in any concrete sense. Early in the next century, two people used zero in a way that transformed the theory of equations. The first of these was Thomas Harriot (1560–1621), a man of many different talents, interests, and accomplishments. Among other things, in 1585 he was sent by Sir Walter Raleigh to help establish the ill-fated Virginia colony on Roanoke Island, now part of North Carolina. He was their surveyor, and he chronicled the settlers' activities and the natural resources of the area.[12] Harriot proposed a simple but powerful technique for understanding algebraic equations:

> Move all the terms of the equation to one side of the equal sign,
> so that the equation takes the form $[\textsf{some polynomial}] = 0$.

This use of zero and negatives unified and simplified the way in which polynomial problems are expressed. In particular, al-Khwārizmī's six cases of a quadratic are no longer separate problems. With negatives and zero available as coefficients, they can all be expressed as particular cases of a general quadratic equation. This more

[12]Harriot's life would make an excellent topic for student projects, particularly in conjunction with other subjects they are studying. It could be used to connect math with geography, history, and natural science.

unified approach also made it easy to see how the coefficients of an equation are related to its roots.

The activities on this sheet show how that notational flexibility leads to more efficient ways of solving quadratic equations. That paves the way for a derivation of the Quadratic Formula in its general form, on Sheet 3-4.

Solutions

1. This preliminary question provides a bit of general historical perspective. All but two of these things occurred in the 16th century. (Students may need to be reminded that the 16th century is the 1500s.)
   (a) 16th. Martin Luther posted his famous *Ninety-Five Theses* on the door of the castle church in Wittenberg, Germany, on October 31, 1517. This is generally regarded as the act that began the Protestant Reformation.
   (b) 16th. Leonardo da Vinci (1452–1519) painted the *Mona Lisa* late in his life. He began it around 1503, but did not complete it until 1519.
   (c) Before. Gutenberg invented movable-type printing in Germany about 1450. The Gutenberg Bible, the first major book printed with movable type in Europe, was finished around 1454.
   (d) 16th. Henry VIII (1491–1547) reigned from 1509 to 1547.
   (e) 16th. Shakespeare was born in 1564.
   (f) After. The Pilgrims landed on the Massachusetts coast late in 1620.
   (g) 16th. The heliocentric theory of Nicolaus Copernicus (1473–1543) was published just before his death in 1543.

2. Besides reinforcing Harriot's Principle, these questions give students some practice in manipulating algebraic expressions and signed numbers.
   (a) $-4x^2 + 8x - 4 = 0$ or $4x^2 - 8x + 4 = 0$
   (b) $2x^3 - 4x^2 - 8x + 12 = 0$ or $-2x^3 + 4x^2 + 8x - 12 = 0$

3. This very easy exercise underscores the flexibility gained by considering negatives and zero as legitimate numbers. The student examples will vary, of course, but some things should be present in all of them. In particular, all the student examples in al-Khwārizmī's form should use only positive numbers. That will then require the use of negatives and zeroes in the same places for all student examples of a given form. We have chosen arbitrary numbers for the answers below; the positions for 0 and negatives are underlined because they are independent of whatever specific numbers students might choose.

   squares equal things: $3x^2 = 5x \quad 3x^2 + (\underline{-}5)x + \underline{0} = 0$

squares equal a number: $5x^2 = 9$ $\quad 5x^2 + \underline{0}x + (\underline{-}9) = 0$

things equal a number: $4x = 7$ $\quad \underline{0}x^2 + 4x + (\underline{-}7) = 0$

squares and things equal a number: $x^2 + 2x = 3$ $\quad 1x^2 + 2x + (\underline{-}3) = 0$

squares equal things and a number: $2x^2 = 5x + 8$ $\quad 2x^2 + (\underline{-}5)x + (\underline{-}8) = 0$

squares and a number equal things: $3x^2 + 1 = 6x$ $\quad 3x^2 + (\underline{-}6)x + 1 = 0$

4. This question illustrates how Harriot's Principle and Descartes' geometry[13] combine to make a powerful tool for approximating solutions to equations of all sorts. It also gives students practice with estimating and refining their approximations, a valuable skill in all applications of mathematics. There is no single "best" answer. Here is a sequence of reasonable approximations, assuming that the tick marks on the x-axis are 1 unit apart. (*Note:* If you let students use calculators to do the routine computations, they will be better able to focus on the main idea.)

   $x = 2.5$; difference: 0.25 $\;(2.5^2 + 11 = 17.25;\ 7 \cdot 2.5 = 17.5)$
   $x = 4.5$; difference: 0.25 $\;(4.5^2 + 11 = 31.25;\ 7 \cdot 4.5 = 31.5)$

   $x = 2.4$; difference: $16.8 - 16.76 = 0.04$
   $x = 4.6$; difference: $32.2 - 32.16 = 0.04$

   $x = 2.38$; difference: $16.6644 - 16.66 = 0.0044$
   $x = 4.62$; difference: $32.3444 - 32.34 = 0.0044$

5. (a) $y = x^4 - 1.4x^3 - 3x^2 + 2x + 1$ (A solution to the equation occurs where this function equals 0.)

   (b) It has four solutions bacause the graph crosses the x-axis in four places.

   (c) Students will need scrap paper for this. They can use calculators to help with the arithmetic of checking their estimates. If they have graphing calculators, however, you might want to change the instructions for this part to have them find the zeroes precisely and plug them back into the original equation to check that they are correct solutions. The following answers are based on "eyeball" estimates of the four zero points on the x-axis of the graph.

   $x = -1.4$ Check: $(-1.4)^4 + 2(-1.4) + 1 \approx 2.042$ and
   $1.4(-1.4)^3 + 3(-1.4)^2 \approx 2.038$. Difference ≈ 0.004

   $x = -0.35$ Check: $(-0.35)^4 + 2(-0.35) + 1 \approx 0.315$ and
   $1.4(-0.35)^3 + 3(-0.35)^2 \approx 0.307$. Difference ≈ 0.008

   $x = 0.9$ Check: $(0.9)^4 + 2(0.9) + 1 = 3.4561$ and
   $1.4(0.9)^3 + 3(0.9)^2 = 3.4506$. Difference $= 0.0055$

[13] We are glossing over a minor anachronism here. Descartes' graphing system did not actually use a fixed y-coordinate, but the general idea was there.

$x = 2.25$ Check: $(2.25)^4 + 2(2.25) + 1 \approx 31.129$ and $1.4(2.25)^3 + 3(2.25)^2 \approx 31.134$. Difference ≈ 0.005

6. This question takes a little careful thought. It might be a good small-group exercise.

   (a) Its zeroes are the values for which $2x^2 - 1 = 7x + 3$. (Harriot's Principle at work.)

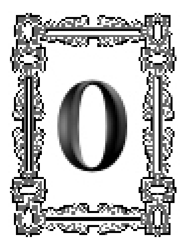
   (b) The zeroes are exactly -0.5 and 4.

   (c) $-0.5 < x < 4$. $2x^2 - 1 < 7x + 3$ if and only if $2x^2 - 1 - (7x + 3) < 0$. But the left side is the function y, so the inequality holds whenever y is negative, which occurs between the two roots in this case.

Sheet 3-3: Using Zero

• Main Feature •
Solving equations by factoring

Sheet 3-2 showed how Harriot's Principle can be used along with coordinate geometry to find approximate solutions to quadratic equations. This sheet looks at how it is used to find exact solutions algebraically.



The power of Harriot's Principle as an equation-solving tool stems from a special property of zero in our number system:

> If the product of two numbers equals zero,
> then at least one of them must be zero.

That is,

if $ab = 0$, then either $a = 0$ or $b = 0$ (or both).

What is impossible with ordinary numbers might be possible in other situations. For example, it is easy to find two nonzero matrices whose product is the zero matrix. Such things are called "zero-divisors." The point here is that in the real number system there are no zero-divisors. At the end of this activity sheet, students will work with a simple number system that has zero-divisors.

Solutions

1. This question makes students notice the fact that there are no zero-divisors. The "why not" part does not require a specific answer. Depending on the grade

and ability level of your students, you might expect an informal response based on repeated addition: Nonzero numbers added together a nonzero number of times can never be zero, or something of that sort. More sophisticated students might justify it by cancellation or by "dividing out" one nonzero number to show that the other must be zero. Responses like these are good because they display an understanding of the basic idea.

2. These are examples of the cancellation law. Students should come to realize that nonzero numbers can be cancelled, but 0 cannot.
   (a) $a = b$ (b) $a = b$ (c) nothing

3. This question presumes that students have some experience with manipulating simple equations and with being asked to justify or explain what they do.

   $3a = 3b$ implies $3a - 3b = 0$, so $3(a - b) = 0$. By #1, $3 \neq 0$ implies $a - b = 0$, so $a = b$.

The observation about the converse of the "no zero-divisors" statement provides on opportunity for two follow-up activities. We have not pursued them because they are not central to the focus of this activity sheet. However, you might find them useful to pursue in class, depending on the requirements of your curriculum.

The first is simply to pursue the idea of *converse* by asking students to state the converse of a statement and then to give an example showing that a statement may be true and its converse false, or vice versa. For example, you might ask them for the converse of "If n is divisible by 4, then it is even." Assuming (or checking) that they know this statement is always true, ask them for an example to show that its converse may be false. It can be useful to try everyday examples, such as "if a bird is a raven, then it is black" or "if this month is August, then it has 31 days."

The second is a bit more sophisticated. The cancellation law and the statement that there are no zero-divisors are actually *equivalent* in a logical sense; that is, either one can be used to prove the other. The solution to #3 shows how cancellation follows from the "no zero-divisors" property. The reverse argument (cancellation implies no zero-divisors) is a good exercise in reasoning, like this:

> Suppose a and b are numbers such that $ab = 0$ and $a \neq 0$.
> Then $ab = a \cdot 0$, so, by cancellation, $b = 0$.
> That is, any product that equals 0 has at least one zero factor.

4. (a) $4x^2 - 8x + 4 = 0 \quad x^2 - 2x + 1 = 0$
   (b) $2x^3 - 4x^2 - 6x + 12 = 0 \quad x^3 - 2x^2 - 3x + 6 = 0$
   (c) $5x^2 - 2x - 7 = 0 \quad x^2 - \frac{2}{5}x - \frac{7}{5} = 0$
   (d) If you factor out a nonzero constant and the product equals 0, then the other factor (the reduced polynomial) must equal 0. (This means that a solution of the simpler equation is also a solution of the original one.)

5. $x^2 - 4x + 3 = (x-3)(x-1) = 0$ is true if either factor equals 0, so both $x-3=0$ and $x-1=0$ give solutions. That is, $x=3$ and $x=1$ are solutions. Check: $3^2 - 4\cdot 3 + 3 = 9 - 12 + 3 = 0$ and $1^2 - 4\cdot 1 + 3 = 1 - 4 + 3 = 0$.

6. This activity presumes that your students know something about factoring quadratic polynomials. In practice, "factoring by inspection" (or by trial and error) is not very useful and so should not be emphasized as much as it has been in some curricula. However, students should learn to recognize some basic forms, including the difference of squares (e.g., x^2-4) and the square of a simple binomial (e.g., $(x+1)^2$.) These quadratics reinforce that recognition.

   (a) $(x+4)(x-1)=0$; $x=-4, 1$

   (b) $4(x-3)(x+3)=0$; $x=3, -3$

   (c) $x(x-11)=0$; $x=0, 11$

   (d) $2(x+5)^2=0$; $x=-5$ (only one solution)

   (e) $(x-4)(x-1)=0$; $x=4, 1$

7. (a) $x(x+1)^2=0$; $x=0, -1$

   (b) $x^3(x-5)(x+5)=0$; $x=0, 5, -5$

   (c) $(x^2+4)(x-2)(x+2)=0$; $x=2, -2$ There are also two complex roots that can be skipped over, unless your students already know about complex numbers. If they do, you might point out that the first factor here can be rewritten as $x^2-(-4)$, which also a difference of squares, $(x-2i)(x+2i)$, giving us the two complex solutions, $2i$ and $-2i$.

As we have already seen, the statements "If a or b equals 0, then $ab=0$" and "If $ab=0$, then a or b equals 0" (no zero-divisors) are converses of each other. There are arithmetic systems in which the first of these statements holds, but the second does not. That is, there are systems in which the basic operations $+$, $-$, $\times$, and $\div$ have most of the same properties as they have in the system of integers, *except* that there are zero-divisors. That makes a huge difference, particularly in solving equations. The rest of the activities on this sheet explore one such system. Not only does this provide a striking example of the operative distinction between a statement and its converse, but it also gives students a much deeper understanding of what it means to solve an equation.

8. The system of integers modulo 12 is sometimes called "clock arithmetic" or "remainder arithmetic mod 12." Any remainder-arithmetic system with a non-prime modulus has zero-divisors; those with a prime modulus do not. We have chosen 12 here because (a) it matches a common clock, and (b) the many factors of 12 provide a lot of examples of zero-divisors.

(a) $5+9=2$; $7+8=3$; $11+1=0$; $3+4=7$; $10+11=9$

(b) $4\times 5=8$; $7\times 3=9$; $6\times 4=0$; $3\times 1=3$; $3\times 5=3$. (The third answer shows that 6 and 4 are zero-divisors in this system.)

(c) Cancellation doesn't work. $3\times 1=3\times 5$, but $1\neq 5$.

9. Because there are zero-divisors, cancellation (and hence, division) does not always work. There may be more or fewer solutions than you would expect in our usual system. Since there are only twelve numbers in the system, an acceptable strategy for finding all solutions is trying all twelve possibilities in each case.

   (a) $x=2$ or 8

   (b) $x=1,5,7,$ or 11

   (c) $x=0$ only

   (d) $x=1,3,5,7,9,$ or 11

10. No. 9 is a solution for $2x-6=0$, but not for $x-3=0$.

Note: Some students might object to the clock arithmetic example of zero-divisors as being too artificial and/or simple. If you have discussed 2×2 matrices in your class, you can use their usual multiplication as a larger, more sophisticated example. For instance,

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \times \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

is a product of two nonzero matrices that equals the zero matrix.

Sheet 3-4: A Method That Always Works

• MAIN FEATURE •

Understanding the Quadratic Formula

This sheet derives the Quadratic Formula from al-Khwārizmī's geometric method of completing the square. It might be helpful (though not essential) to have students look back at their work on sheet 3-1 before doing this sheet. As the title of this sheet suggests, the importance of the Quadratic Formula stems from the fact that it can be used to solve any quadratic equation.

#5 touches on the issue that sometimes the solutions involve square roots of negatives. The history of the gradual acceptance of complex numbers is a major theme of the next chapter.

Solutions

1. This is a "reminder" exercise in how literal-constant notation works.
   (a) Correct answers will vary. Look for x and y remaining as letters (unknowns) in every example. Also look for the same number for both appearances of the same letter. Typical examples: $2x^2+3x-4=4y+2$; $3x^2+5x-8=8y+5$; $9x^2+2x-7=7y+9$.
   (b) $ax-bx^2+c=dy^3+ey$
   (c) This part highlights the fact that a constant letter represents the same number every time it appears in an expression, and different letters *may* (or may not) stand for different numbers. The given polynomial does not fit the second because its leading coefficient and its constant term are different ($5\neq 9$). However, $5x^2+8x+5$ (or any other quadratic with its constant term the same as its leading coefficient) fits both forms because c is allowed to equal a.

2. $x^2+\left(\frac{b}{a}\right)x=\frac{-c}{a}$ The answers must be positive because they represent a length and an area. (The coefficient of x is a length; the constant term is an area.) From al-Khwārizmı's viewpoint, this means that $\frac{-c}{a}$ may be positive, too. That is, "the negative of" c is a positive number.

3. This activity derives the Quadratic Formula by the "completing the square" geometrically. As on Sheet 3-1, the boxes inside the various regions represent areas; the outside boxes represent lengths. Question #4 will observe that the formula works even when the quantities are not positive.
   (a) The bottom lengths are x and $\frac{b}{a}$. The rectangular area is $\frac{b}{a}\cdot x$.
   (b) Each bottom length is $\frac{b}{2a}$, so each strip area is $\frac{b}{2a}x$.
   (c) Each strip area is still $\frac{b}{2a}x$; the length at left is still $\frac{b}{2a}$.
   (d) Each strip area is still $\frac{b}{2a}x$; the area of the missing square is $(\frac{b}{2a})^2$.
   (e) This is the part that pulls the derivation together. The area of the region is $x^2+\frac{b}{a}x$ or $\frac{-c}{a}$. (That's what the quadratic equation says.) It is the same as the region in (d) without the missing corner square. Therefore, the area of the completed square is $\frac{-c}{a}+(\frac{b}{2a})^2$. Its side length is $x+\frac{b}{2a}$, so $x=\sqrt{\frac{-c}{a}+(\frac{b}{2a})^2}-\frac{b}{2a}$.

(f) $\sqrt{\dfrac{-c}{a}+\left(\dfrac{b}{2a}\right)^2}-\dfrac{b}{2a} = \sqrt{\dfrac{-c}{a}+\dfrac{b^2}{4a^2}}-\dfrac{b}{2a}$

$$= \sqrt{\frac{-4ac+b^2}{4a^2}}-\frac{b}{2a} = \frac{\sqrt{b^2-4ac}}{2a}-\frac{b}{2a} = \frac{-b+\sqrt{b^2-4ac}}{2a}$$

4. This provides good practice in manipulating algebraic expressions. (If you are willing to let your students simply accept the fact that these solutions always work, this exercise can be skipped.) The verification for the positive root works like this:

$$a\left(\frac{-b+\sqrt{b^2-4ac}}{2a}\right)^2+b\left(\frac{-b+\sqrt{b^2-4ac}}{2a}\right)+c$$
$$= a\left(\frac{b^2-2b\sqrt{b^2-4ac}+b^2-4ac}{4a^2}\right)+\frac{-b^2+b\sqrt{b^2-4ac}}{2a}+c$$
$$= \frac{2b^2-2b\sqrt{b^2-4ac}-4ac}{4a}+\frac{-b^2+b\sqrt{b^2-4ac}}{2a}+c$$
$$= \frac{b^2-b\sqrt{b^2-4ac}-2ac-b^2+b\sqrt{b^2-4ac}+2ac}{2a} = \frac{0}{2a} = 0$$

The verification for the negative root is similar.

This is a striking example of what the English algebraists came to call "the generality of algebra." The computation was done on the assumption that $b > 0$ and $c < 0$. Once the answer is written as an algebraic formula, however, you can check that the formula gives a solution as an algebraic identity. Once you know that, you no longer need to care about the signs of b and c.

5. This question foreshadows the discussion of imaginary numbers that will appear in the next chapter. Student opinions probably will vary, but they might guess correctly that mathematicians of that time generally accepted Descartes' view of which numbers were *real*, in the literal sense of that word. Hence, since the square of any (real) number must be positive, the square root of a negative was "unreal" — that is, not legitimate.

Once they could graph the parabola, this case was a little less disturbing because when $b^2-4ac < 0$, there is no intersection with the x-axis. Therefore, the negative number inside a square root could just be read as "there is no solution."

6. (a) $x = \dfrac{-8\pm\sqrt{8^2-4\cdot 3\cdot(-17)}}{2\cdot 3} = \dfrac{-8\pm\sqrt{268}}{6}$, so $x \approx 1.395$ or -4.06. Students can check their answers in several different ways. If they have graphing calculators, they can enter the quadratic function and ask the calculator to find its zeroes. Otherwise, they can substitute either the exact values (with the radicals) or the approximations and work out the arithmetic to get 0 either exactly or approximately, respectively.

(b) This part reinforces the fact that the Quadratic Formula works for *any* quadratic equation. If you are using this activity sheet in a class setting, you might reinforce this point further by having several students present the equations they chose.

Sheet 3-5: Quadratics in Earlier Times

• MAIN FEATURE •
Analyzing mathematical arguments

These activities are somewhat more challenging than most of the ones on the previous sheets in this chapter. They will stretch your students' algebraic thinking and help them to see how multi-step solutions may be structured. All of them involve relating the Quadratic Formula to earlier ways of solving quadratic equations.

Students will need scrap paper for some of these activities.

Solutions

1. (a) If $x+y=20$, then $y=20-x$, so $x^2+(20-x)^2=208$. This simplifies to $x^2-20x+96=0$. These numbers are "nice" enough for the equation to be factored by inspection. $96=12\cdot 8$, so $x^2-20x+96=(x-12)(x-8)=0$. Therefore, $x=12$ or $x=8$.

   (b) As in part (a), if $x+y=a$, then $y=a-x$, so $x^2+(a-x)^2=b$. This simplifies to $2x^2-2ax+(a^2-b)=0$ or $x^2-ax+\frac{a^2-b}{2}=0$.

   (c) The notation can be a trap for the unwary here. The a and b in this equation are not the same as the a and b in the Quadratic Formula. This sort of thing is common mathematical practice, so students need to be flexible in interpreting the notation of formulas in unfamiliar situations.

$$x=\frac{-(-a)\pm\sqrt{(-a)^2-4(\frac{a^2-b}{2})}}{2}=\frac{a\pm\sqrt{2b-a^2}}{2}$$

   (d) In the notation of part (c), this means $2b-a^2$ is a square, making $\sqrt{2b-a^2}$ a rational number. "Is it really necessary?" challenges student understanding of the history of numbers. In Diophantus's time, the only acceptable numbers were rational. Therefore, the square root in this case had to come out rational, so it was necessary *in the mathematics of the third century* (the time of Diophantus). Eighteenth-century mathematicians would have required only that "double the sum of the squares exceed the square of the sum" because they would not allow the square root of a negative. Today the restriction is unnecessary; $2b - a^2$ negative simply means that the roots are complex.

2. The statement of this problem is from Martin Levey, *The Algebra of Abū Kāmil, in a commentary by Mordecai Finzi*, University of Wisconsin Press, 1966, page 156. The student work in this part uses modern algebraic notation and the Quadratic Formula, neither of which was available in Abū Kāmil's time. That made this a much more difficult problem then!

   (a) $\left(\dfrac{v}{w}\right)^2 - \left(\dfrac{w}{v}\right)^2 = 2$

   (b) Students must first recognize that the larger quotient is $\frac{v}{w}$ because $v > w$. Then the equation of (a) becomes $x^2 - \dfrac{1}{x^2} = 2$, which can be rewritten as $(x^2)^2 - 1 = 2x^2$ or $x^4 - 2x^2 - 1 = 0$.

   (c) By the Quadratic Formula, $x^2 = \dfrac{2 \pm \sqrt{4+4}}{2} = \dfrac{2 \pm 2\sqrt{2}}{2} = 1 \pm \sqrt{2}$. The positive answer is $x^2 = 1 + \sqrt{2}$.

   (d) $x = \frac{v}{w} = \frac{10-w}{w}$. Thus, $(\frac{10-w}{w})^2 = 1 + \sqrt{2}$, so $(10-w)^2 = (1+\sqrt{2})w^2$. This simplifies to $\sqrt{2}w^2 + 20w - 100 = 0$. By the Quadratic Formula, $w = \frac{-10 \pm 10\sqrt{1+\sqrt{2}}}{\sqrt{2}}$. Simplifying and taking only the positive result, we get $w = \frac{10}{\sqrt{2}}(\sqrt{1+\sqrt{2}} - 1)$.

   (e) This shows students how to make an approximate check on the reasonableness of a messy-looking answer. Using a calculator, they should get $w \approx 3.9$, so $v \approx 10 - 3.9 = 6.1$. Therefore,

   $$\left(\frac{v}{w}\right)^2 - \left(\frac{w}{v}\right)^2 \approx \left(\frac{6.1}{3.9}\right)^2 - \left(\frac{3.9}{6.1}\right)^2 \approx 2.04$$

   The error is less than 0.04 for a one-place approximation of the exact answer, so that answer seems reasonably likely to be correct.

3. This problem is adapted from p. 31 of [8]. Students may find it quite difficult to translate the scribe's wording into comprehensible modern language. A detailed explanation of the Babylonian sexagesimal system can be found in Sketches 1 and 4 of [2].

(a) $4;45 = 4 + \frac{45}{60} = 4\frac{3}{4}$
$3,20;30 = 3 \cdot 60 + 20 + \frac{30}{60} = 200\frac{1}{2}$
$1,2;10,20 = 1 \cdot 60 + 2 + \frac{10}{60} + \frac{20}{3600} = 62 + \frac{1}{6} + \frac{1}{180} = 62\frac{31}{180}$

(b) $25\frac{1}{4} = 25;15 \qquad 63\frac{1}{2} = 1,3;30 \qquad 245\frac{3}{4} = 4,5;45$

(c) The sum of the area and the side of a square equals $\frac{45}{60}$ (which is $\frac{3}{4}$). What is the length of the side?

(d) Equation: $x^2 + x = \frac{3}{4}$ or $x^2 + x - \frac{3}{4} = 0$ Solution: $x = \frac{1}{2}$
Using the Quadratic Formula will help students with the next questions.
$x = \dfrac{-1 \pm \sqrt{1+3}}{2} = \dfrac{-1+2}{2} = \dfrac{1}{2}$ (The negative root is not considered.)

(e) $x = \dfrac{-b \pm \sqrt{b^2 - 4c}}{2} = \dfrac{-b}{2} \pm \sqrt{\dfrac{b^2 - 4c}{4}} = \dfrac{-b}{2} \pm \sqrt{\left(\dfrac{b}{2}\right)^2 - c}$

(f) *You write down 1, the coefficient.* This is b.
You break off half of 1: This is $\frac{b}{2} = \frac{1}{2}$.
0;30 and 0;30 you multiply: 0;15. $(\frac{b}{2})^2 = (\frac{1}{2})^2 = \frac{1}{4}$
You add 0;15 to 0;45: 1. $(\frac{b}{2})^2 + c = \frac{1}{4} + \frac{3}{4} = 1$.
This is the square of 1. Its square root equals 1.
From 1 you subtract 0;30... $1 - \frac{1}{2}$ (This is $\left(\sqrt{\left(\frac{b}{2}\right)^2 - c}\right) - \frac{b}{2}$.)
0;30 $= \frac{1}{2}$ is the side of the square.

4

Cubics and Imaginaries

Mathematical Focus	**Third-Degree Equations**
Historical Connections	Greece, 3rd century BCE to 5th century CE India, 10th century Middle East, 11th century Italy, 16th century France & Germany, 17th–18th centuries

The value of algebra in a high school education is directly proportional to its effectiveness in enhancing the student's ability to think clearly and reason effectively. For most students, knowing how to manipulate polynomial equations, especially those of degree 3 and higher, is far less important than understanding the underlying ideas. The Common Core State Standards for Mathematics (CCSSM) reflect the importance of this conceptual understanding in several places.

For instance, one of the high school algebra standards is:

> Understand solving equations as a process of reasoning and explain the reasoning.

The Standards for Mathematical Practice, which cut across all topics in the mathematics curriculum, include these:

- Make sense of problems and persevere in solving them.
- Reason abstractly and quantitatively.
- Construct viable arguments and critique the reasoning of others.
- Look for and express regularity in repeated reasoning.

This chapter is ideally suited to implementing these standards. It traces the pursuit of a very difficult problem — finding a general solution for cubic equations — from its roots in Greek mathematics to its final resolution in 16th-century Italy. It examines the reasoning as algebra gradually became more symbolic and less geometric, and it shows how simplifying a truly difficult question can provide steppingstones to a final solution. Along the way, a more abstract understanding of *number* evolved, leading to the complex number system.

Mathematical problems rarely arise in abstract form. The problem of solving equations of degree three grew out of geometric problems first considered by ancient Greek mathematicians. The original problems go back as far as 400 BCE or so, but their complete solutions only came some 2000 years later. The quest for those solutions was, in turn, the motivation for constructing the complex numbers, which provide a powerful link between algebra and the geometry of the plane.

Sheet 4-1: Boxes, Lines, & Angles

• MAIN FEATURE •
The geometry of cubic equations

This sheet illustrates how geometry and algebra are intertwined by exploring two geometric problems that translate into cubic equations. They are the first two of these three famous "Problems of Antiquity" that originally arose as compass-and-straightedge construction challenges in ancient Greek times:

- Find the side of a cube with volume twice that of a given cube. (*Duplication of the Cube*)
- Divide a given angle into three equal parts. (*Trisection of the Angle*)
- Find a square with area equal to that of a given circle. (*Quadrature of the Circle*)

Already in Greek times it was known that the first two problems are easier than the third. They can be solved by finding the intersection points of two conics, by sliding a marked ruler on a diagram, and by using various mechanical tools built for the purpose. Squaring the circle was clearly much harder.

From a modern point of view, the first two problems require the solution of a cubic equation, while squaring the circle requires a construction of a line segment of length π. In this activity sheet we address only the first two problems. (π is a *transcendental* number, which means that it is not the solution of any polynomial equation with rational coefficients.)

Solutions

1. The first problem in the list above is also known as the Delian Problem because of this legend about the Oracle of Apollo at Delos. According to the legend, when the Athenians realized their error and appealed to Plato for help, he told them that the oracle gave them this problem "to reproach the Greeks for their neglect of mathematics and their contempt of geometry."[14]

   (a) In this context, it is something that speaks for a god, someone or something that speaks with supernatural authority.

   (b) They doubled the scale (the linear measure), not the "size" (the volume measure). Of course, the oracle's directive was a bit ambiguous, but that was often the way with oracles. It was commonly accepted that the pronouncements of an oracle might come in the form of a riddle or some other obscure statement which would have to be interpreted with great care lest it be misunderstood.

   (c) The oracle wanted them to construct a cube with twice the *volume* of the original altar.

   (d) There are two "natural" ways to do this, depending on what is chosen as the known quantity. If students choose the volume of the original altar as the known constant, say V, then $x^3 = 2V$, so $x = \sqrt[3]{2V}$. If they think of the edge length of the original altar as the known constant, say a, then $x^3 = 2a^3$, so $x = \sqrt[3]{2} \cdot a$. The Greeks probably would have used the latter form (in words) because it does not mix lengths and volumes, which to them were different types of quantities.

   (e) This is a good question for class discussion. There are several reasons why this was difficult for the Greeks. Some should be fairly obvious; others are more subtle.

      - To begin with, the form of the question would have been completely in words; algebraic symbols, as we know them, did not exist then.
      - Not only were there no calculators, there wasn't even an algorithm for finding cube roots by hand!
      - Less obvious, but more to the point, is the fact that from the point of view of Greek mathematicians what was needed was a precise (not approximate!) geometric construction of the a line segment that would serve as the edge of the cube.

[14] [3], p. 119.

- The Greeks thought of lengths and volumes as different *kinds* of quantities; combining them required special care.

2. Hippocrates of Chios (c. 470–410 BCE) is not the same as Hippocrates of Cos (c. 460–370 BCE), the physician from whom we get the Hippocratic Oath.

   (a) Rewrite the two equalities as equations in x and y (by cross-multiplication), then substitute for y and solve for x, as follows: $8y = x^2$ and $27x = y^2$, so $27x = (\frac{x^2}{8})^2 = \frac{x^4}{64}$; that is, $27 \cdot 64 = x^3$, so $x = 3 \cdot 4 = 12$. Then $8y = 12^2$, so $y = \frac{144}{8} = 18$. Check: $\frac{8}{12} = \frac{12}{18} = \frac{18}{27}$.

   (b) This mirrors the computation in part (a), with a and $2a$ in place of 8 and 27, respectively. $ay = x^2$ and $2ax = y^2$, so $2ax = (\frac{x^2}{a})^2 = \frac{x^4}{a^2}$; that is, $2a^3 = x^3$, so $x = \sqrt[3]{2} \cdot a$. This is the same as the 1(d) equation for students who used edge length (a), rather than volume (V). For those who used volume, a simple substitution of $V = a^3$ results in this equation.

3. This activity introduces some elementary trigonometry as a way to translate questions about angles into algebraic equations. The questions themselves, however, require only algebraic manipulation. Students can think of $\sin\alpha$, $\sin\beta$, $\cos\alpha$, and $\cos\beta$ as four arbitrary symbols — say, a, b, c, and d — and work out the required calculation. These answers are used as an essential part of the next activity.

   (a) $(\sin\alpha)^2 + (\cos\alpha)^2 = 1$, one of the most fundamental identities in right-angle trigonometry. You might use this as an opportunity to tell your students that the usual way to write the squares of trigonometric values is to put the exponent on the function, like this: $\sin^2\alpha + \cos^2\alpha = 1$.

   (b) $\cos 2\alpha = \cos(\alpha + \alpha)$, so substitute α for β in the angle-sum formula to get $\cos 2\alpha = \cos^2\alpha - \sin^2\alpha$ [or $\cos 2\alpha = (\cos\alpha)^2 - (\sin\alpha)^2$].

   (c) Substitute as in part (b):
   $\sin 2\alpha = \sin(\alpha + \alpha) = \sin\alpha \cdot \cos\alpha + \cos\alpha \cdot \sin\alpha = 2\sin\alpha\cos\alpha$.

   The answers to (b) and (c) are the *double angle formulas* for sine and cosine.

4. In modern language, we would say that the cosine function is one-to-one on angles between 0° and 90°. This is a multi-step calculation that will require much more space than that on the activity sheet. The first step is perhaps the one that requires the most thought. To use the machinery of #3, rewrite $\cos 3\alpha$ as $\cos(2\alpha+\alpha)$. Then the rest is a matter of making careful substitutions and simplifying, as follows:

$$\begin{aligned}\cos(2\alpha + \alpha) &= \cos 2\alpha \cdot \cos\alpha - \sin 2\alpha \cdot \sin\alpha \\ &= (\cos^2\alpha - \sin^2\alpha)\cos\alpha - (2\sin\alpha \cdot \cos\alpha)\sin\alpha\end{aligned}$$

$$
\begin{aligned}
&= \cos^3\alpha - \sin^2\alpha\cdot\cos\alpha - 2\sin^2\alpha\cdot\cos\alpha\\
&= \cos^3\alpha - 3\sin^2\alpha\cdot\cos\alpha\\
&= \cos^3\alpha - 3(1-\cos^2\alpha)\cos\alpha\\
&= \cos^3\alpha - 3\cos\alpha + 3\cos^3\alpha\\
&= 4\cos^3\alpha - 3\cos\alpha
\end{aligned}
$$

Therefore, $\cos\theta = 4x^3 - 3x$.

5. This is just the Pythagorean Theorem at work.
   (a) Since the cosine is the side of a right angle with hypotenuse 1, the third side can be found by the Pythagorean Theorem. Since all three sides of the triangle are known, all the angles are determined, by SSS congruence. (That is, only one triangular shape has these three side lengths.)
   (b) By #3(a), $\sin^2\theta + \cos^2\theta = 1$. Therefore, $\sin^2\theta = 1 - \cos^2\theta = 1 - 0.82^2 = 0.3276$, so $\sin\theta = \sqrt{0.3276} \approx 0.57$.
   (c) The slope measure of an angle θ is its "rise over run": $\dfrac{\sin\theta}{\cos\theta} = \dfrac{0.57}{0.82} \approx 0.7$. Trigonometry calls this the *tangent* of the angle.

Sheet 4-2: From Shapes to Numbers

• MAIN FEATURE •
Writing cubic equations algebraically

This sheet begins quite gently. Historically, it looks at the conceptual shift that unhooked numbers from their geometric interpretations. Pedagogically, it provides a good example of abstraction as a simplifying process. Once numbers were regarded as mathematical "things" on their own, without being tied to real-world quantities of some sort, the use of negatives and zero became much more acceptable. This greatly simplified the way to write equations.

This conceptual shift went hand in hand with another, perhaps more profound one. As the use of equations became more widespread, the focus moved from concrete problems to the equations themselves. Things that earlier looked very different, such as a geometric problem and a puzzle about numbers, turned out to boil down to the same equation. Once people realized their importance, equations, like numbers, began to be studied as objects in their own right, independent of any real-world context, and classified according to their own patterns, such as their degree.

Eventually the many different types of cubic equations studied in the 11th century became special cases of the general cubic equation that we use today.

Solutions

1. (a) (i) $x^3 + 3x = 14$ (ii) $2x^3 = 3x + 10$ (iii) $x^3 + 5 = 26x$

   (b) This is an exercise in number sense; no algorithm is needed.
   sample: $x = 3$ (i) $x = 2$ (ii) $x = 2$ (iii) $x = 5$

2. (a) His most famous work is the Rubaiyat, which means "quatrains." It is a book of romantic poetry, (freely) translated in 1859 by Edward FitzGerald and published as *Rubáiyát of Omar Khayyám*. Perhaps the most famous quatrain in FitzGerald's book is this:

   A Book of Verses underneath the Bough,
   A Jug of Wine, a Loaf of Bread — and Thou
   Beside me singing in the Wilderness —
   Oh, Wilderness were Paradise enow!

   (b) Khayyam and his contemporaries knew that these forms could be reduced to quadratics. They didn't have our notation for "dividing out" the common factor x, but they understood the concept.

   (c) This part reinforces part (b) and also the fact that 0 and negatives were not acceptable roots. The calculations also provide some routine algebra practice. Using modern notation, each of these reduces to a quadratic that can be factored and solved by inspection.

   $x^3 = 4x$ becomes $x(x^2 - 4) = 0$, so $x(x + 2)(x - 2) = 0$.
   Roots: $0, -2, 2$ Khayyam's roots: 2 (only)

   $x^3 + 3x^2 = 4x$ becomes $x(x^2 + 3x - 4) = 0$, so $x(x + 4)(x - 1) = 0$.
   Roots: $0, -4, 1$ Khayyam's roots: 1 (only)

   $x^3 + 2x = 3x^2$ becomes $x(x^2 - 3x + 2) = 0$, so $x(x - 2)(x - 1) = 0$.
   Roots: $0, 1, 2$ Khayyam's roots: 1 and 2

3. (a) The choice of letters to represent the coefficients and constant terms is unimportant, and can vary from student to student. We have chosen to use them in alphabetical order here, regardless of the degree of the term. Notice that the plurals of *squares* and *numbers* signal coefficients; that is, "some squares" and "some numbers."

   $x^3 + ax^2 = b$ A cube and squares equal a number.

   $x^3 + a = bx^2$ A cube and a number equal squares.

   $x^3 + ax = b$ A cube and roots equal a number.

$x^3 = ax + b$ A cube equals roots and a number.

$x^3 = ax^2 + b$ A cube equals squares and a number.

(b) Because he could always divide through by any other coefficient to make the leading coefficient 1.

(c) This reflects the general principle of part (b).

$3x^3 = 2x^2 + 12$ becomes $x^3 = \frac{2}{3}x^2 + 4$.

$5x^3 + 8x = 3$ becomes $x^3 + \frac{8}{5}x = \frac{3}{5}$

$(\frac{3}{4})x^3 + 12x^2 = 5$ becomes $x^3 + 16x^2 = \frac{20}{3}$.

4. This is largely a "bookkeeping" exercise. Its main purpose is to reinforce the idea that follows — namely, that a having single form for all cubic equations is *much* more convenient than dealing with all these separate types. Using the literal coefficients in alphabetical order, the other forms are:

   $x^3 = ax^2 + bx + c$

   $x^3 + ax = bx^2 + c$

   $x^3 + a = bx^2 + cx$

   $x^3 + ax^2 + bx = c$

   $x^3 + ax^2 + b = cx$

   $x^3 + ax + b = cx^2$.

5. This highlights the main idea of this activity sheet: The combination of standard algebraic notation and a more abstract, non-geometric idea of *number* provide a simpler, more unified way of looking at cubic equations (and many other kinds of equations, too). This leads to a much more effective approach to solving equations.

   (a) $x^3 + (0)x^2 + (6)x + (-7) = 0$

   (b) $x^3 + (-4)x^2 + (-5)x + (-1) = 0$

   (c) $x^3 + (-2)x^2 + (0)x + (3) = 0$

   (d) $x^3 + (-8)x^2 + (1)x + (-9) = 0$

   (e) $x^3 + (0)x^2 + (-19)x + (0) = 0$

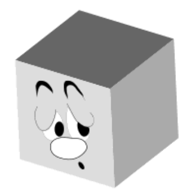
   (f) $x^3 + (0)x^2 + (-\frac{4}{3})x + (-\frac{3}{2}) = 0$

   (g) $x^3 + (5)x^2 + (-24)x + (11) = 0$

Sheet 4-3: The Depressed Cubic

• MAIN FEATURE •
Solving some cubic equations

Algebra reached Italy in the 13th century, when Leonardo of Pisa's *Liber Abbaci* introduced both algebra and arithmetic with Hindu-Arabic numerals. In the following centuries, a lively tradition of arithmetic and algebra teaching developed in Italy, as the Italian "abbacists" tried to meet the increasing computational needs of the growing merchant class. Luca Pacioli's 1494 *Summa de Aritmetica*, a very influential book about the arithmetic and algebra known at that time, described how to solve linear and quadratic equations. However, Pacioli was not hopeful that a general method for solving cubic equations could be found.



The crucial breakthrough was made in Italy, first by Scipione del Ferro (1465–1526) and then by Tartaglia[15] (1500–1557). Both men discovered how to solve certain kinds of cubic equations. Both men kept their solutions secret, because at this time scholars were mostly supported by rich patrons and had to earn their jobs by defeating other scholars in public competitions. Knowing how to solve cubic equations allowed them to challenge the others with problems that they knew the others could not solve, so people were inclined to keep quiet about their discoveries.

In the case of the cubic, this pattern was broken by Girolamo Cardano (1501–1576). Promising never to reveal it, Cardano convinced Tartaglia to share the secret of the cubic with him. Once he knew Tartaglia's method for solving some cubic equations, Cardano was able to generalize it to a way of solving any cubic equation. Feeling that he had actually made a contribution of his own, Cardano decided that he was no longer bound by his promise of secrecy and published the method in his book, *Ars Magna*. (For a fuller account of this story of personal intrigue and manipulation, See Ch. 6 of [7] or Sketch 11 of [2].)

The activities on this sheet give students some practice manipulating algebraic expressions involving powers and roots as they follow the Tartaglia/Cardano recipe for solving "depressed cubics."

Note: Students who go on to do Activity Sheets 4-4 and 4-5 will need to use some of their results on this sheet as they work through those activities.

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[15]Though this mathematician's real name was Niccolò Fontana, everyone knows him by his nickname, Tartaglia, which means "stammerer."

Solutions

1. Note that, as mentioned in Chapter 1, *thing* refers to the unknown. In this context, it is a synonym for *root*.

   (a) $x^3 + ax = b$

   (b) The x-coefficients and constant terms will vary, but the form needs to be the same.

   (c) A cube equals some things and a number. $x^3 = ax + b$
   A cube and a number equal some things. $x^3 + a = bx$
   (Students may use different letters for the coefficients and constant terms.)

2. (a) (Actually, Cardano wrote this as "cubus p̃. 6. rebus aequalis 20.")
   Solve for x: $x^3 + 6x = 20$

   (b) $2^3 + \left(\frac{1}{2} \cdot 20\right)^2 = 108$, so we have $\sqrt{108}$.

   (c) $\sqrt{108} + 10$ and $\sqrt{108} - 10$

   (d) $x = \sqrt[3]{\sqrt{108} + 10} - \sqrt[3]{\sqrt{108} - 10}$

   (e) Surprisingly, $x = 2$. Check: $2^3 + 6 \cdot 2 = 8 + 12 = 20$

3. $\sqrt{4^3 + 324} = \sqrt{388}$
   $\sqrt{388} + 18$ and $\sqrt{388} - 18$
   $x = \sqrt[3]{\sqrt{388} + 18} - \sqrt[3]{\sqrt{388} - 18} \approx 2.16$
   A calculator check using the calculator's exact answer will come out exactly right. If 2.16 is used, the calculator will return a constant term of approximately 35.998, correct within the roundoff error.

4. The letters used are not significant. We chose c for the coefficient of x and k as the constant merely as a mnemonic aid. This result is needed for #2 and #4 of Sheet 4-4 and #5 of Sheet 4-5.
   $$\sqrt{\left(\frac{c}{3}\right)^3 + \left(\frac{k}{2}\right)^2} = \sqrt{\frac{c^3}{27} + \frac{k^2}{4}}$$
   $$\sqrt{\frac{c^3}{27} + \frac{k^2}{4}} + \frac{k}{2} \text{ and } \sqrt{\frac{c^3}{27} + \frac{k^2}{4}} - \frac{k}{2}$$
   $$\boxed{x = \sqrt[3]{\sqrt{\frac{c^3}{27} + \frac{k^2}{4}} + \frac{k}{2}} - \sqrt[3]{\sqrt{\frac{c^3}{27} + \frac{k^2}{4}} - \frac{k}{2}}}$$
   Sometimes doing the coefficient divisions makes this formula easier to use:
   $$x = \sqrt[3]{\sqrt{\left(\frac{c}{3}\right)^3 + \left(\frac{k}{2}\right)^2} + \frac{k}{2}} - \sqrt[3]{\sqrt{\left(\frac{c}{3}\right)^3 + \left(\frac{k}{2}\right)^2} - \frac{k}{2}}$$

5. (a) $x = \sqrt[3]{\sqrt{27+169}+13} - \sqrt[3]{\sqrt{27+169}-13}$
 $= \sqrt[3]{14+13} - \sqrt[3]{14-13} = 3 - 1 = 2$
 Check: $2^3 + 9 \cdot 2 = 8 + 18 = 26$

 (b) $x = \sqrt[3]{\sqrt{5}+2} - \sqrt[3]{\sqrt{5}-2}$
 This messy expression turns out to be equal to 1, but it's not obvious! One way to check is to evaluate it with a calculator, and then check that 1 is a solution to the equation. (To be absolutely sure that this is not some other real solution within calculator roundoff error of 1, you could graph $y = x^3 + 3x - 4$ and see that it only crosses the x-axis once in that neighborhood of 1. Or you could factor out $x - 1$ and observe that the resulting quadratic had no real roots. Of course, Cardano didn't have such nice tools to work with!)

6. These coefficients have been chosen so that the results are easy to check. Nevertheless, the arithmetic is best done with a calculator.

 (a) Rearranged: $x^3 + (-21)x = 90$
 Solution: $x = \sqrt[3]{\sqrt{(-7)^3 + 45^2}+45} - \sqrt[3]{\sqrt{(-7)^3 + 45^2}-45}$
 $= \sqrt[3]{\sqrt{1682}+45} - \sqrt[3]{\sqrt{1682}-45} = 6$
 Check: $6^3 = 216$ and $21 \cdot 6 + 90 = 126 + 90 = 216$

 (b) Rearranged: $x^3 + (-6)x = -40$
 Solution: $x = \sqrt[3]{\sqrt{(-2)^3 + (-20)^2}+(-20)} - \sqrt[3]{\sqrt{(-2)^3 + (-20)^2}-(-20)}$
 $= \sqrt[3]{\sqrt{392}-20} - \sqrt[3]{\sqrt{392}+20} = -4$
 Check: $(-4)^3 + 40 = -64 + 40 = -24$ and $6(-4) = -24$

Sheet 4-4: The General Cubic

• Main Feature •
Solving any cubic equation

This sheet exemplifies the way in which mathematicians extend and generalize the solution of a special case of a problem to get to find a general solution. It typifies one of George Pólya's famous problem-solving tips, "Solve a Simpler Problem," a way to resolve a problem that seems very difficult if handled all at once. In this case, Cardano extended Tartaglia's solution of the depressed cubic by finding a way to convert *any* cubic into a depressed cubic.

The algebraic manipulations in these activities are a bit more challenging than those of previous sheets. They provide good practice for motivated students, but might not be suitable for students who struggle with symbolic calculation.

Solutions

1. This is a somewhat tedious exercise in algebraic manipulation, but there is a worthwhile payoff at the end.

   (a) $\left(y - \frac{a}{3}\right)^2 = y^2 - \frac{2a}{3}y + \frac{a^2}{9}$ Now multiply this by $y - \frac{a}{3}$ and simplify:
   $\left(y - \frac{a}{3}\right)^3 = y^3 - ay^2 + \frac{a^2}{3}y - \frac{a^3}{27}$

   (b) This part requires some persistence and attention to detail.
   $$\left(y^3 - ay^2 + \frac{a^2}{3}y - \frac{a^3}{27}\right) + a\left(y^2 - \frac{2a}{3}y + \frac{a^2}{9}\right) + b\left(y - \frac{a}{3}\right) + c = 0$$
   $$y^3 + (0)\,y^2 + \left(b - \frac{a^2}{3}\right)y + \left(c + \frac{2a^3 - 9ab}{27}\right) = 0$$

   (c) ...because there is no y^2 term (i.e., the coefficient is 0).

2. (a) $y = x + \frac{6}{3} = x + 2$, so $x = y - 2$.
   $(y-2)^2 = y^2 - 4y + 4$
   $(y-2)^3 = y^3 - 6y^2 + 12y - 8 = 0$

   (b) $(y^3 - 6y^2 + 12y - 8) + 6(y^2 - 4y + 4) + 10(y - 2) + 8 = 0$
   $y^3 - 2y + 4 = 0$

   (c) Students will need their results from Sheet 4-3. Some care is needed to avoid confusing the variable symbols. This is good practice for students who might do later work in math or science. Notation often varies from one source to another; it is the reader's job to make the appropriate adjustments. In the form of #4 of Sheet 4-3, this equation is $y^3 - 2y = -4$, so the solution is
   $$y = \sqrt[3]{\sqrt{\left(\frac{-2}{3}\right)^3 + \left(\frac{-4}{2}\right)^2} + \frac{-4}{2}} - \sqrt[3]{\sqrt{\left(\frac{-2}{3}\right)^3 + \left(\frac{-4}{2}\right)^2} - \frac{-4}{2}}$$
   $$= \sqrt[3]{\frac{10}{\sqrt{27}} - 2} - \sqrt[3]{\frac{10}{\sqrt{27}} + 2}.$$ Calculator evaluation: $y = -2$

   (d) $x = y - 2 = -4$
   Check: $(-4)^3 + 6(-4)^2 + 10(-4) + 8 = -64 + 96 - 40 + 8 = 0$

3. Just divide all terms by the leading coefficient. This yields an equivalent equation with leading coefficient 1, so Cardano's formula can be used.

4. Besides providing some algebraic practice, this activity illustrates why square roots of negative numbers cannot simply be ignored.

   (a) $3^3 + 18 = 27 + 18 = 45 = 15 \cdot 3$

   (b) To use Cardano's formula (from Sheet 4-3), start by putting this equation into his depressed cubic form: $x^3 - 15x = -18$. Then $c = -15$ and $k = -18$, so

   $$x = \sqrt[3]{\sqrt{\left(\frac{-15}{3}\right)^3 + \left(\frac{-18}{2}\right)^2} + \frac{-18}{2}} - \sqrt[3]{\sqrt{\left(\frac{-15}{3}\right)^3 + \left(\frac{-18}{2}\right)^2} - \frac{-18}{2}}$$

   $$x = \sqrt[3]{\sqrt{-44} - 9} - \sqrt[3]{\sqrt{-44} + 9}$$

   (c) It involves the square root of a negative number, which Cardano would have regarded as meaningless.

   (d) This part presupposes that students have learned how to divide a linear factor out of a polynomial. If that is not something they have studied, you may want to give them the factored form.

   Begin by rewriting the equation as $x^3 - 15x + 18 = 0$. Then divide the left side by $x-3$, to get $x^3 - 15x + 18 = (x-3)(x^2+3x-6)$. Thus, the other two solutions come from solving $x^2+3x-6 = 0$ by the Quadratic Formula: $x = \dfrac{-3 \pm \sqrt{33}}{2}$. The important thing to observe here is that *both of the solutions are real numbers.* That is, even though Cardano's formula involved square roots of negatives, three *real* solutions exist. Thus, square roots of negatives can't simply be dismissed as meaningless.

5. This activity helps students see the development of algebra as part of the broad cultural awakening of the Renaissance.

   Michelangelo Buonarroti (1475–1564); art (sculptor, painter, architect). His most famous works include the sculptures *Pietà* and *David*, the painted frescoes of the Sistine Chapel, and the architectural design of the dome of St. Peter's Basilica at the Vatican.

   Giovanni **Palestrina** (c. 1525–1594); music (composer of sacred music). No single work is as well-known as the artwork of Michelangelo, but students might come up with the names of two of his 104 masses, *Missa Papae Marcelli* and *Missa sine nomine* ("mass without a name"). Students who do serious choral singing might know some of his other works.

   Raphael is Raffaello Sanzio (1483–1520); art (painter and architect). One of many Raphael masterpieces is the *Madonna* featured on a 2011 U.S. Christmas stamp. (A different *Madonna* by Raphael is on a 1973 Christmas stamp.)

Galileo Galilei (1564–1642); science, or "natural philosophy" (astronomer and physicist). He is best known for his promulgation of the Copernican theory that the Earth revolved about the Sun, a position for which he was condemned and put under house arrest by the Roman Catholic Church. Among his best-known writings are *On the Revolution of the Heavenly Orbs* and *Discourses Concerning the Two New Sciences.*

6. This activity emphasizes the clarity and efficiency of algebraic notation. It also serves as a caution not to trust everything you read.[16]

   (a) $x = \sqrt[4]{d + \left(\frac{c}{a}\right)^2} - \sqrt{\frac{c}{a}}$

   (b) The coefficient of x^2 is never used! (If it worked, changing the x^2 term would never change the solution, which is nonsense.)

Sheet 4-5: Impossible, Imaginary, Useful

• MAIN FEATURE •
Imaginary and complex numbers

The standard way of introducing the complex numbers is to argue that we want to be able to solve all quadratic equations, including $x^2 + 1 = 0$. The obvious reaction to that is: "Why?" It's a good question. During many centuries of the study of algebraic equations, mathematicians thought of them as a means for solving concrete problems about length, distance, weight, etc. In this context, even negative solutions didn't make much sense. And if applying the quadratic formula led you to the square root of a negative number, this simply meant that your problem had no solution.

It was not quadratics, but *cubics*, that pushed mathematicians to deal constructively with square roots of negatives. Cardano's formula for solving cubic equations sometimes led to square roots of negatives even when there was a positive real solution! This paradox led some 16th-century algebraists to take square roots of negatives more seriously, devising formal rules for manipulating them, even though they doubted their "real" existence.

[16] This example is taken from Section 4.4 of [11].

This sheet explores the historical motivation for imaginary numbers, the basis for complex numbers. This historical foundation provides the gateway to a smooth transition into as full a presentation of the complex number system as you might choose to present. These activities also give students a meaningful context in which to exercise their manipulative algebra skills. In that respect, some of them are a bit more challenging than the questions on previous sheets.

The story of complex numbers can also be used to emphasize to your students that new mathematical ideas only become important when they become useful. As long as people were working only with quadratic equations, they did not need to learn to work with complex numbers. If they ran into a square root of a negative number, it was reasonable to say "the problem does not have a solution" and be done with it. (#2 in this activity sheet is an example of this.) It was only when they were confronted with a problem that did have a solution, but whose solution required complex numbers, that these new numbers began being treated seriously.

Solutions

1. This is just a quick reminder about how squares and square roots work.

   (a) $5^2 = 25$ (b) $(-5)^2 = 25$ (c) $(\sqrt{5})^2 = 5$

   (d) positive (e) positive (f) can't tell

2. This activity is more challenging. It requires some careful, creative thought.

   (a) The product of two numbers, say x and y, is the area of an x-by-y rectangle. Students may know from earlier work, or may infer by some exploratory calculations, that the largest such area is a square, where $x = y$. In this case, that means the largest value for xy is 25, so $xy = 40$ would be impossible.

   (b) Students need to begin by converting the conditions into a single quadratic equation. $x + y = 10$ and $xy = 40$, so $y = 10 - x$ and $x(10 - x) = 40$. That is, $x^2 - 10x + 40 = 0$. By the Quadratic Formula,

   $$x = \frac{10 \pm \sqrt{100 - 160}}{2} = 5 \pm \sqrt{-15}$$

   If $x = 5 + \sqrt{-15}$, then $y = 10 - (5 + \sqrt{-15}) = 5 - \sqrt{-15}$, and vice versa.

   (c) Students need to do what Cardano did — "suspend disbelief" about square roots of negatives and calculate with them as if the "obvious" rules of arithmetic make sense for them.

   $(5 + \sqrt{-15}) + (5 - \sqrt{-15}) = 5 + 5 = 10$, and

   $(5 + \sqrt{-15}) \cdot (5 - \sqrt{-15}) = 25 - (\sqrt{-15})^2 = 25 + 15 = 40$

Cardano dismissed this kind of thing, saying it is "as refined as it is useless," i.e., it is very sophisticated and very useless, a meaningless intellectual game. In other words, Cardano is telling us that there is no need to study complex numbers, since they only arise from problems without a solution.

In another book, Cardano seemed more positive, though not less confused, saying that "$\sqrt{9}$ is either $+3$ or -3, for a plus [times a plus] or a minus times a minus yields a plus. Therefore $\sqrt{-9}$ is neither $+3$ or -3 but is some recondite [obscure] third sort of thing."[17]

3. (a) They appear where the graph (of the expression on the left) crosses the x-axis.

   (b) Graphing calculators or computers can be used for this without compromising the point of the question. Recall from #2(b) that the equation is $x^2 - 10x + 40 = 0$. The graph of $x^2 - 10x + 40$ does not cross the x-axis anywhere. Thus, the formula asks for the square root of a negative number exactly when the equation has no (real) solutions. That's as expected: when the formula fails, it is because there is no (real) answer.

4. This may be quite difficult for students, depending on how much experience they have had with graphing polynomials. The basic intuitive idea is this: While the graphs of quadratics turn once (are parabolas), the graphs of cubics turn twice. This means that a cubic always crosses the x-axis somewhere, which is a real solution for it. Therefore, a cubic equation will always have at least one real solution.

5. (a) Cardano's formula, as it appeared on sheet 4-3 for the solution of an equation of the form $x^3 + cx = k$, is

$$x = \sqrt[3]{\sqrt{\frac{c^3}{27} + \frac{k^2}{4}} + \frac{k}{2}} - \sqrt[3]{\sqrt{\frac{c^3}{27} + \frac{k^2}{4}} - \frac{k}{2}}, \text{ or}$$

$$x = \sqrt[3]{\sqrt{\left(\frac{c}{3}\right)^3 + \left(\frac{k}{2}\right)^2} + \frac{k}{2}} - \sqrt[3]{\sqrt{\left(\frac{c}{3}\right)^3 + \left(\frac{k}{2}\right)^2} - \frac{k}{2}}$$

To use it properly, students first have to rearrange $x^3 = 15x + 4$ into the appropriate form, $x^3 - 15x = 4$. Then

$$\begin{aligned} x &= \sqrt[3]{\sqrt{\left(\frac{-15}{3}\right)^3 + \left(\frac{4}{2}\right)^2} + \frac{4}{2}} - \sqrt[3]{\sqrt{\left(\frac{-15}{3}\right)^3 + \left(\frac{4}{2}\right)^2} - \frac{4}{2}} \\ &= \sqrt[3]{\sqrt{(-5)^3 + 2^2} + 2} - \sqrt[3]{\sqrt{(-5)^3 + 2^2} - 2} \end{aligned}$$

[17] From *Ars Magna Arithmeticae*, problem 38, quoted in [5], p. 220, note 6.

$$= \sqrt[3]{\sqrt{-121}+2} \;-\; \sqrt[3]{\sqrt{-121}-2}, \text{ which can also be written as}$$

$$x \;=\; \sqrt[3]{2+\sqrt{-121}} \;+\; \sqrt[3]{2-\sqrt{-121}}$$

This result is used in the next activity.

(b) $64 = 4^3 = 15 \cdot 4 + 4 = 64$

(c) This is an exercise in polynomial division. It can be done by synthetic division, of course, but need not be. If your students haven't been taught long division of polynomials, they can do this by mimicking long division of numbers. (If they haven't learned how to do that, you might have to give them this answer.)

$$\begin{array}{r@{}l}
 & \;\; x^2 + \;4x + 1 \\ \cline{2-2}
x-4 \,\big)\! & \; x^3 \qquad\quad - 15x - 4 \\
-\,(& \underline{x^3 - 4x^2}) \\
 & \qquad 4x^2 - 15x \\
 & \quad -\,(\underline{4x^2 - 16x}) \\
 & \qquad\qquad\quad x - 4
\end{array}$$

That is, $x^3 - 15x - 4 \;=\; (x-4)\cdot(x^2+4x+1)$

(d) $x = \dfrac{-4 \pm \sqrt{4^2-4}}{2} = \dfrac{-4 \pm \sqrt{12}}{2} = -2 \pm \sqrt{3}$

They are both real. Cardano's "imaginary" solution turned up for this equation, but it has three real roots, and no imaginary ones! (This sort of thing will always happen when the equation has three real roots.)

(e) Students may have to adjust the Window parameters a bit to get a good picture. The graph should show that both the cubic and the quadratic cross the negative x-axis at the same two points.

6. (a) $(2+\sqrt{-1})^2 = 4 + 4\sqrt{-1} - 1 = 3 + 4\sqrt{-1}$

$(3+4\sqrt{-1})(2+\sqrt{-1}) = 6 + 11\sqrt{-1} - 4 = 2 + 11\sqrt{-1}$

$(2-\sqrt{-1})^2 = 4 - 4\sqrt{-1} - 1 = 3 - 4\sqrt{-1}$

$(3-4\sqrt{-1})(2-\sqrt{-1}) = 6 - 11\sqrt{-1} - 4 = 2 - 11\sqrt{-1}$

(b) Students need to notice that $11\sqrt{-1} = \sqrt{-121}$; then everything else is pretty easy.

$$\begin{aligned}
x \;&=\; \sqrt[3]{2+\sqrt{-121}} \;+\; \sqrt[3]{2-\sqrt{-121}} \\
&=\; \sqrt[3]{(2+\sqrt{-1})^3} \;+\; \sqrt[3]{(2-\sqrt{-1})^3} \\
&=\; 2+\sqrt{-1}+2-\sqrt{-1} \;=\; 4
\end{aligned}$$

7. This easy exercise illustrates the transition to the now-standard notation for imaginary and complex numbers. The use of i appears to have originated with Euler in the late 1700s and was made popular by Gauss early in the next

century. The French mathematician Cauchy used it extensively, citing it as the usage of the German mathematicians.

(a) $\sqrt{-25} = 5\sqrt{-1}$ $\quad$ $\sqrt{-100} = 10\sqrt{-1}$ $\quad$ $\sqrt{-5} = \sqrt{5}\sqrt{-1}$
$\sqrt{-12} = \sqrt{12}\sqrt{-1} = 2\sqrt{3}\sqrt{-1}$

(b) $5i$ $\quad$ $10i$ $\quad$ $\sqrt{5}i$ $\quad$ $2\sqrt{3}i$

This notational improvement made the one-to-one correspondence between reals and imaginaries quite clear. That, in turn, provided the key to interpreting the entire system of complex numbers as the points on a plane.

If your curriculum includes a more extensive look at the system of complex numbers, it is easy to move right into that from here. A possible sequence of concepts is as follows:

- Adding imaginaries to the real number system prompts the need for closure under addition (and subtraction). This is what requires the complex form $a + bi$. (Gauss named such things *complex numbers* in the early 1800s.)
- The four arithmetic operations — $+$, $-$, $\times$, and $\div$ — need to be defined in a way that respects $i^2 = -1$ and results in something of the same complex form. Of these, the most difficult one is division.
- Since multiplying numbers by -1 can be visualized as flipping the number line $180°$ around 0, it's not unreasonable to envision multiplication by $\sqrt{-1}$ as a $90°$ rotation. This leads to the representation of the complex numbers as the points on a plane, with $a + bi$ corresponding to the point (a, b).
- Linear order is not possible, but the plane provides a rich visual representation of the system. Interpreting the complex numbers as vectors in the plane leads to many possible avenues of further investigation, beginning with a visual sense of how the arithmetic operations work and moving on to powers and roots.
- The great bonus is that we do not just acquire roots for *some* polynomial equations, we end up with roots for *all* of them. It turns out that, if we allow complex numbers, then every polynomial equation will have roots.

Notes

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Activity Sheet Set

for

Pathways from the Past

II: Using History to Teach Algebra

William P. Berlinghoff
Fernando Q. Gouvêa

Oxton House Publishers
2013

Format Note: The activity sheets in this set are two-sided. That is reflected in the page numbering and in the border for each pair of pages. If you print this as a two-sided document, the sheets will come out the same as in the original print version.

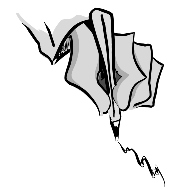
Name:________________________________ Date:______________

Activity 1-1

Symbols of Arithmetic

1. Use arithmetic symbols to write

   "When 9 is subtracted from the sum of 4 and 7, the result is 2."



   ____

   Do you think this expression is clearer in words or in symbols?

   [ ] in words [ ] in symbols

   Why ?____

   ____

The symbols you used to answer question 1 are universal. A student in Brazil or China or Egypt could read your answer and know exactly what you meant. That wasn't always so. Those symbols are only a few centuries old. The ancient Greeks and early Arab scholars wrote arithmetic in words. In fact, arithmetic and algebra were written entirely in words by most people for many, many centuries, right through the Middle Ages.

2. Write each of these equations in words.
   (a) (5 + 6) – 7 = 4 ____

   ____

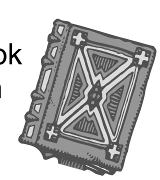
   (b) 24 – (9 + 6) = 10 – 1 ____

   ____

During the Renaissance, people began to study more mathematics to help them understand the world around them. Writing down their ideas by hand was slow. Even movable-type printing, invented in the 1440s, was a tedious way to put ideas on paper. Some mathematicians began to make up symbols for common mathematical ideas and processes.

3. In the 1470s, the famous German scientist Regiomontanus would have written (4 + 7) – 9 = 2 as 4 *et* 7 ℘ 9 — 2 . (The word *et* is Latin for *and*.)
   (a) How would he have written (5 + 6) – 7 = 4? ____________________________
   (b) How would he have written 12 – 5 = 4 + 3 ? ____________________________
   (c) Write 17 ℘ 6 *et* 4 — 12 *et* 8 ℘ 5 in our symbols. ________________________

4. In 1494, Italian mathematician Luca Pacioli published a large reference book of practical mathematics, known as *Summa de Aritmetica*. The symbols in this book became very common throughout much of 16th-century Europe. Pacioli would have written (4 + 7) – 9 = 2 as 4 $\tilde{p}$ 7 $\tilde{m}$ 9 — 2 .



   (a) How would he have written 12 – 5 = 4 + 3 ? ____________________________
   (b) Translate 12 $\tilde{p}$ 28 $\tilde{m}$ 9 — 25 $\tilde{p}$ 6 into modern symbols.______________________

5. To indicate grouping, writers of the late 15th and early 16th centuries often used underlining, not parentheses, such as $9\ \tilde{p}\ \underline{5\ \tilde{m}\ 3}$ for 9 + (5 – 3).

   (a) Write $\underline{9\ \tilde{m}\ 5}\ \tilde{m}\ \underline{3\ \tilde{p}\ 4}$ — $4\ \tilde{p}\ 7$ in our notation.________________________________

   (b) Calculate: $27\ \tilde{p}\ \underline{13\ \tilde{m}\ 6}\ \tilde{m}\ \underline{11\ \tilde{p}\ 4}\ \tilde{m}\ \underline{3\ \tilde{p}\ 6}$ — ______

   (c) True or false: $120\ \tilde{m}\ \underline{46\ \tilde{m}\ 17}$ — $120\ \tilde{m}\ 46\ \tilde{p}\ 17$. __________

6. The first time the symbols + and – appeared in a book written in English was in Robert Recorde's 1557 algebra text, *The Whetstone of Witte*. He also introduced = for equality there. All his symbols were elongated. For instance, he would have written 4 + 7 – 9 = 2 as 4 —+— 7 —— 9 ═══ 2 .

   (a) Write 68 – 23 = 40 + 5 his way. ________________________________

   (b) Calculate: 17 —— 5 ═══ 9 —+— __ .

Recorde's notation did not catch on right away. In fact, most European writers continued to use $\tilde{p}$ and $\tilde{m}$ or similar symbols for another century or so. In the 17th and 18th centuries, ÷ was often used for subtraction, too.

7. In the 1630s, William Oughtred in England used +, –, and = in a book that emphasized the importance of mathematical symbols. He used colons for grouping. For instance, (4 + 7) – 9 = 2 would be **:**4 + 7**:** – 9 = 2.

   (a) Calculate: 24 – **:**5 + 7**:** – 2 + **:**8 – 3**:** = ____

   (b) True or false: **:**18 + 10**:** – 7 = 18 + **:**10 – 7**:** _________

   (c) True or false: **:**18 – 10**:** + 7 = 18 – **:**10 + 7**:** _________

8. In 1637, the French mathematician René Descartes's book *La Géométrie* simplified much of the notation of that time. His fame led to widespread acceptance of his symbols. Most but not all of them were like what we use today. For subtraction he used a broken dash, and for equality he used a strange symbol: $\backpropto$. Grouping was denoted by dots (periods). He would have written 12 – (5 + 3) = 4 like this:

   12 -- **.**5 + 3**.** $\backpropto$ 4

   (a) Calculate: 18 -- **.**5 + 6**.** + 3 -- 2 $\backpropto$ ____

   (b) Calculate: 37 + 5 -- **.**14 + 6**.** -- **.**8 + 4**.** $\backpropto$ ____

   (c) True or false: 20 -- 3 -- **.**12 + 4**.** $\backpropto$ 1 _________

   (d) True or false: 15 -- **.**9 + 5**.** $\backpropto$ 15 -- 9 -- 5 _________

   (e) True or false: **.**40 -- 25**.** -- 10 $\backpropto$ 40 -- **.**25 -- 10**.** _________

9. What was going on in North America during the first half of the 17th century? Mention one or two notable things.

____

____

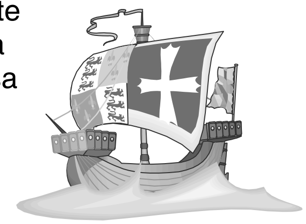
____

Name:_______________________________ Date:_____________

Activity 1-2

Algebra in Italy, 1200–1550

An **algebra problem** is a question about finding an unknown quantity from known ones. Until only a few hundred years ago, algebra was done entirely in words, with no symbols at all.



1. At the beginning of the 13th century, Pisa was an important city-state on the northwest coast of Italy. It had its own naval fleet and was a major center of trade and commerce. At that time, Leonardo of Pisa would have written an algebra problem like this:

   Five more than the square of a thing is equal to fourteen.
   What is the thing?

   (a) Solve this problem without writing it in symbols. __________

   (b) Write this problem in algebraic symbols. ____________________

   (c) Write the following problem entirely in words, in the style of the example above:

   $$x^2 - 3 = 4x + 2; \quad x = ?$$

   ____

   ____

   (d) Solve the problem in (c). ____________________

2. Write these algebra equations in symbols.

   (a) Four more than twice the square of a thing is equal to six times the thing.

   ____

   (b) Ten times a thing is equal to three more than the cube of the thing.

   ____

   (c) The square of one less than a thing equals the double of three more than the thing. ____

   (d) For each one, find a (positive) thing that works *without using algebra symbols*.

   a: ___________ b: ___________ c: ___________

3. Write these three equations in words, as Leonardo of Pisa might have done.

   (a) $4x + 7 = 25 - x$ ____

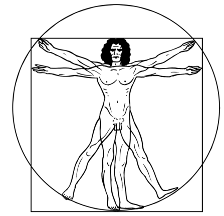
   ____

   (b) $2x^2 - 1 = 5x + 3$ ____

   ____

(c) $x^3 - 2x^2 = 10x + 9$ ____

____



Three centuries after Leonardo of Pisa, Italy had progressed from the Middle Ages to the Renaissance. Art, music, commerce, and learning flourished together, particularly in cities such as Florence and Venice. The need for reliable math in business led to books on arithmetic and algebra written in everyday Italian, rather than Latin. The best known one was Luca Pacioli's 1494 book, *Summa de Aritmetica*.

4. If you were studying algebra in southern Europe around 1500, you likely would have used Pacioli's book. Pacioli was one of the first people to use abbreviations and symbols in algebraic expressions.

   He would have written $x^3 - 2x^2 = 12x + 7$ like this:

$$1.cu.\tilde{m}.2.ce.\text{—}12.co.\tilde{p}.7.$$

   (a) *co* abbreviates *cosa* (the unknown "thing") and *ce* abbreviates *censo* (its square). What do the other symbols stand for?

   cu _________ $\tilde{m}$ _________ $\tilde{p}$ _________ — _________

   (b) What do you think the dots are for? ________________________________

   (c) Write $1.\tilde{p}.7.ce.\text{—}2.cu.\tilde{m}.1.co.$ in modern notation. ______________________

   (d) Write $5x^3 + 3x = x^2 - 2$ in Pacioli's notation. ________________________

5. Like many others of his time, Pacioli used the Latin word *radix* ("root" or "base") for the square root of a number. Its symbol was ℞; he wrote ℞.200. for $\sqrt{200}$. To take the square root of more than one term, Pacioli added the letter v (for *universalis*), meaning that everything after that was to be included. For example:

$$\text{℞}\,v.1.co.\tilde{p}.36. \text{ meant } \sqrt{x+36}$$

   Write in modern symbols:

   (a) $\text{℞}v.9.\tilde{p}.\text{℞}.49.\text{—}\,4.$ _____________ Is it correct?______

   (b) $\text{℞}v.4.ce.\tilde{m}.12.co.\tilde{p}.9.\text{—}\,2.co.\tilde{m}.3.$ __________________ Is it correct? ______

   Write in symbols that Pacioli would have understood:

   (c) $\sqrt{9} + \sqrt{16} = \sqrt{50-1}$ _________________________ Is it correct? ______

   (d) $\sqrt{x^2 - 4x + 4} = x - \sqrt{4}$ ____________________________ Is it correct? ______

6. Solve each of these problems any way you can.

   (a) Write $2.cu.\tilde{m}.3.ce.\tilde{m}.\text{℞}\ 4.co.$ as a product of linear factors in modern notation. __________________________________

   (b) Write the product of $3.ce.\tilde{p}.5.$ and $4.co.\tilde{m}.1.$ in Pacioli's notation. _________________________________

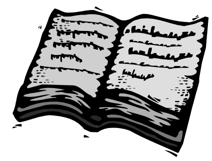
   (c) $9.ce.\tilde{p}.2.co.\tilde{m}.16.\text{—}\,\text{℞}v.4.ce.$ has a positive solution. Find it. co — _____

Name:__________________________________ Date:______________

Activity 1-3

Germany and France, 1450–1600

Algebra in 16th-century Germany looked very different than it did in Italy. If you were studying Christoff Rudolff's widely used algebra book in 1530, you would have written $x^3 - 5x^2 + 7x = \sqrt{x+6}$ like this: ꝯ – 5ȝ + 7ꝛ *aequ.* √.ꝛ + 6.



1. (a) In the equation above, *aequ.* is an abbreviation for *aequetur*, which is Latin for "equals." What do each of these symbols mean?

   ꝯ ____ ȝ ____ ꝛ ____

   (b) Explain how √.ȝ + 2ꝛ + 1. *aequ.* √ȝ + 1 can be true. ____________________

   ____

   ____

   (c) How did Rudolff show that the square root included more than one term?

   ____

   (d) Translate this equation into modern notation; then solve it.

   5ȝ – 20 *aequ.* √.ȝ + 4ꝛ + 4. __________________ ꝛ *aequ.* _________

   (e) Translate into Rudolff's notation: $3x^3 + 5x^2 - \sqrt{7} = 4x + \sqrt{x^3 - x + 6}$

   ____

2. Rudolff and others of his time wrote higher powers of the unknown by combining symbols when they could and inventing new ones when they had to. Their system worked like this:

x^4	ȝȝ	x^7	bß (another new symbol)
x^5	ß (a new symbol)	x^8	ȝȝȝ
x^6	ȝꝯ	x^9	ꝯꝯ

   (a) You can think of ȝȝ as either $x^2 \cdot x^2$ or $(x^2)^2$. Which of the following choices are represented by ȝꝯ ? (Check all that apply.)

   __ $x^2 \cdot x^3$ __ $(x^2)^3$ __ $x^3 \cdot x^3$ __ $(x^3)^2$ __ $x^3 \cdot x^2$ __ none of these

   (b) Does ȝȝȝ represent $x^4 \cdot x^2$ or $(x^4)^2$? ______ How do you know?

   ____

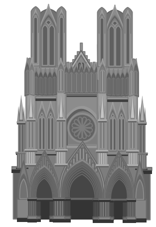
   (c) How do you think Rudolff would write x^{10} ? _______ How about x^{12} ? ______

   (d) Describe the pattern for these symbols. __________________________________

   ____

   (e) Why does x^{11} need a new symbol? _________________________________

Germany and France, 1450–1600



Having a different symbol for each power of the unknown made it hard to understand how different powers were related to each other. In 1484, a French physician named Nicolas Chuquet had taken a big first step toward curing this problem, but his manuscript was not published for a long time, so his innovative work was unknown to most 16th-century Europeans.

3. Much like the 15th-century Italians, Chuquet used $\overline{p}$ and $\overline{m}$ for + and –, and the French word *montent* for "equals." However, he represented powers of the unknown by putting exponents on their coefficients, *with no symbol at all for the unknown itself*. For instance, he wrote $4x^2$ as 4^2, x^5 as 1^5, and so on.

   (a) Write in modern notation:

   $5^4.\overline{m}.1^2.\overline{p}.7.$ *montent* $2^3.\overline{p}.1^1.\overline{m}.1.$ ________________________________

   $1^4.\overline{p}.1^3.\overline{p}.1^2.\overline{p}.1^1.\overline{p}.1.$ *montent* $2^5.\overline{m}.32.$ ________________________________

   (b) Write in Chuquet's notation:

   $x^5 + 3x^2 - 1 = 4x^4 - 2x + 8$ ________________________________

   $35x^2 - 7x + 5 = x^3 - x^2 + 2x + 2$ ________________________________

4. Chuquet also used exponents for roots, and to take a root of more than one term, he underlined the grouping: For instance, $\sqrt[3]{5}$ was $℞^3.5.$ and $\sqrt{x+5}$ was $℞^2.\underline{1^1.\overline{p}.5}.$

   (a) Write in modern notation:

   $℞^2.\underline{1^2.\overline{m}.6^1.\overline{p}.9}.$ *montent* $.1^1.\overline{m}.℞^2.9.$ ________________________________

   $1^4.\overline{p}.5^3.\overline{m}.℞^2.4^1.$ *montent* $.℞^3.\underline{1^2.\overline{p}.℞^2.12^1}.$ ________________________________

   (b) Write in Chuquet's notation:

   $\sqrt{x^2 + 10x + \sqrt{625}} = x + \sqrt{25}$ ________________________________

   $7x^4 - 3x^3 + \sqrt[3]{2x^2 - 9x} + \sqrt[5]{8x - 1}$ ________________________________

   (c) Calculate, using positive roots. Write your answer in Chuquet's notation.

   $℞^2.\underline{4^2.\overline{p}.24^1.\overline{p}.36}.\overline{m}.1^1.\overline{m}.℞^2.16.$ *montent* __________

5. In the 1570s, Italian mathematician Rafael Bombelli used a notation like Chuquet's. He put powers of the unknown in little cups above their coefficients. A few decades later Simon Stevin, a Flemish engineer, adopted this notation in the form of circles around coefficients to denote powers of the unknown. For instance, he wrote $2x^3 + 8x^2 - 24x - 96$ as 2③ + 8② — 24① — 96 .

   (a) Write 3⑤ — 7③ + 9② — 11① in modern notation. ________________________

   (b) Write $6x^4 + x^2 - x + 5$ in Stevin's notation. ________________________

   (c) Stevin used *M* for multiplication. Answer TRUE or FALSE:

   3② *M* 3④ equals 3⑥ ________ 3② *M* 3④ equals 3⑧ ________

   3② *M* 3④ equals 9⑥ ________ 1② *M* 1④ equals 1⑥ ________

Name:________________________________Date:____________

Activity 1-4

Letters for Numbers

A big change in notation came at the end of the 16th century. François Viète, a lawyer and an advisor to King Henri IV of France, also wrote about solving algebraic equations. To write general forms of equations, he used letters for both unknowns and constants. This avoided relying on examples in which the numbers chosen might improperly affect the general solution process. Some earlier writers had experimented with letters, but Viète was the first to use them as an important part of algebra.

1. Viète used vowels for unknowns and consonants for constants. The word "in" was used for multiplication. For instance, he would have written $3x + 5y = 7$ as

3 in A + 5 in E aequ. 7

and its general form (with letters for the constants) as *B in A + C in E aequ. D*.

(a) Write *5 in A + 6 aequ. 4 in E – 2* in modern notation. ____________________

(b) Write the general form of *7 in A + 3 in E aequ. 9* in two ways:

Viète's way: ______________________ Modern way: ___________________

(c) Write $y = mx + b$ in Viète's notation. _______________________

(d) Write two different instances of *B in A – C in E + D aequ. F* in modern notation. Make all the constants in the second instance different from those in the first.

________________________ ___________________________

Viète used words for higher powers of unknowns, such as *A quadratus* and *A cubus* for A^2 and A^3. But early in the 17th century people began to use symbols for powers.

2. In England, Thomas Harriot's 1631 book used lowercase letters for unknowns and just repeated them for higher powers. For instance, Harriot wrote a^3 as *aaa*. He used Robert Recorde's double-line sign for *equals* and a shorter form of his *plus* and *minus* signs. $52 - 7a^2 = 3a + a^3$ was written like this:

$$52 - 7.aa = 3.a + aaa$$

(a) Write $5.aaa + 2.aa - 4.a = 9$ in modern notation. ____________________

(b) Solve: $4.aa - 9 = 0$. ____

(c) True or False: $9.aaaa = 3.aa + 6.aa$ ___________

3. Harriot also invented the symbols $>$ and $<$ for "greater than" and "less than." We use shorter forms of these same symbols today.

(a) Is $aa < aaa$ true for all positive real numbers? ______ Explain.____________

____

(b) For which positive real numbers is $aaa > 7a$? ________________

(c) For which real numbers (including negatives) is $aaaa < aa$?_______________

$2 < 5$ $3 > 1$ $7 < 8 < 9$

4. In Paris in the 1630s, Pierre Hérigone devised a symbol system for powers that avoided the words of Viète and the repeated letters of Harriot. Using lowercase letters for unknowns, he showed the number of repeated copies in a product by putting a number behind the letter. For instance, he wrote $a^3 + 5e^2$ as a3 + 5*e*2.

   Hérigone's system included symbols for geometric things, too, including – for *straight line* and = for *parallel lines*. He used ~ for minus and *est* (Latin for "it is") to express equality.

   (a) Write in modern notation:

   7*a*4 + 2*a*3 ~ *a*2 est 2*a* ~ 1 ____________________________

   *a*5 ~ 3*a*4 ~ *a*3 + 7*a* est 5*a*2*e* + *ae*2 ~ 2*e* ____________________________

   (b) Write in Hérigone's notation:

   $a^4 - 4a^3 + 2a^2 - 3a + 5 = 0$ ________________________________

   $a^3 + 3a^2e + 3ae^2 + e^3 = 2a^2e^2 - 1$ ________________________________

   (c) One of these is correct; the other is not. Which is which?

   2*a*2*e* + 2*ae*2 est 4*a*3*e*3 _________   2*a*2*e*2 + *a*2*e*2 est 3*a*2*e*2 _________

5. James Hume, an Englishman living in Paris, took a big step toward the exponential notation we use today. In his 1636 book about the algebra of Viète, he wrote A *cubus* as A^{iii}. That is, he used small raised Roman numerals as exponents! For *plus* and *minus*, he used + and –, which were becoming standard in France.

   (a) Write $A^{v} - 2A^{iv} + 3A^{iii} - 5A^{ii}$ in modern notation. __________________________

   (b) Write $A^6 + 3A^4 - 5A^3 + 2A^2$ in Hume's notation. _________________________

6. Much of our modern notation comes from René Descartes' *La Géométrie* of 1637. He used the letters *a, b, c, ...* for known quantities (constants) and letters *z, y, x, ...* for unknowns. He also wrote exponents as small raised Hindu-Arabic numerals, except for squares, which he often wrote as a doubled symbol. But his symbols for subtraction and equality did *not* survive. He would have written the general form of $y = x^4 + 2x^3 - 3x^2 + 4x - 5$ as

$$y \propto x^4 + ax^3 \text{ -- } bxx + cx \text{ -- } d.$$

   (a) Write $ax^2 + bx + c = y^3 - 1$ in Descartes' notation. _________________________

   (b) Write $2xx \propto 6x \text{ -- } 4$ in modern notation, and solve it. _______________________

   Solution:____

   ____

7. How would $x^3 - 3x^2y + 3xy^2 = y^3$ have been written by

   (a) students of Thomas Harriot's book in 1631? _________________________________

   (b) Pierre Hérigone in 1634? _________________________________

   (c) James Hume in 1636? (Use *donne* for =.) _________________________________

   (d) René Descartes in 1637? _________________________________

Name:________________________________ Date:______________

Activity 2-1

The Rule of Three Direct

1. You work for 2 hours to help paint a house and are paid $16.

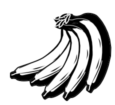
   (a) How much should you be paid for working 4 hours? _____

   (b) How much should you be paid for working 3 hours? _____

   (c) How much should you be paid for working 2 hours and 45 min.? _____

   (d) Explain your answers. ____

   ____



Question 1 starts with a **ratio**, a relationship between two quantities. Ratios often involve different kinds of quantities — dollars and hours, feet and seconds, etc. A **proportion** is a statement that two ratios express the same relationship. If $2 buys 5 bananas and $4 buys 10 bananas, the two costs are *in proportion*. We say, "2 (dollars) is to 5 (bananas) as 4 (dollars) is to 10 (bananas)." The traditional way of writing this proportion is

2 : 5 :: 4 : 10.

2. (a) Write your results for parts (a), (b), and (c) of #1 in proportion notation.

   (1a) _______________ (1b) _______________ (1c) _______________

   (b) Write a real-world example of 4 : 240 :: 6 : 360.____________________________

   ____

   (c) If $3 buys 5 lbs. of potatoes, how much should 15 lbs. cost? _____

   Write this as a proportion. ____________________

Long before symbolic algebra, merchants and traders calculated proportions by the ***Rule of Three***. This rule was taught and learned in the same form for more than 2000 years, from China in 100 BCE to 5th-century India to 13th-century Europe to the United States in the 1800s. Daboll's *Schoolmaster's Assistant* of 1799 says it like this:

> "The Rule of Three Direct teaches, by having three numbers given to find a fourth, which shall have the same proportion to the third, as the second has to the first....
> RULE.—1. [P]lace the numbers so that the first and third terms may be of the same kind; and the second term of the same kind with the answer, or thing sought.
> 2. Bring the first and third terms to the same denomination, and reduce the second term to the lowest name mentioned in it.
> 3. Multiply the second and third terms together, and divide their product by the first term; and the quotient will be the answer to the question, in the same denomination you left the second term in...."

3. Use the Rule of Three Direct to solve #1b and #1c above. Show your work.

   (1b) ______________________________ (1c) ______________________________

   ______________________________ ______________________________

   ______________________________ ______________________________

The Rule of Three Direct

The American colonists of the 1600s and 1700s used many different kinds of money. Two of the most common kinds were British pounds (£) and Spanish dollars (known to 17th-century pirates as "pieces of eight"). There were 8 *bits* in a Spanish dollar, 20 *shillings* (*s*) in a pound and 12 *pence* (*d*) in a shilling. ("Pence" is the British plural of "penny").

<u>Big problem:</u> Each colony controlled the exchange rate for its money. This meant that a pound in Massachusetts might be worth more or less than a pound in New York or Pennsylvania or Virginia. In the 1700s, people doing business in more than one colony had to be very good at figuring out exchange values. The Rule of Three was a big help.

4. These are Rule of Three Direct questions from Nicolas Pike's 1809 *Arithmetick*. Show the Rule of Three setup, and state your answers in pounds, shillings, and pence).

   (a) A merchant delivered at Boston 320£ Massachusetts currency to receive 400£ in Philadelphia; what was the Massachusetts pound valued at?

   (b) My correspondent in Maryland purchased a cargo of flour for me, for 437£ that currency; how much Massachusetts money must I remit him, 125£ Maryland being equal to 100£ Massachusetts?

5. In the late 1700s, the currency exchange rate (in shillings and pence) was

   4*s* 6*d* British = 6*s* Massachusetts = 7*s* 6*d* Pennsylvania = 8*s* New York.

   Show how the Rule of Three Direct can be used to find the value of £1 Pennsylvania in each of the other three places. Write your answer in pounds, shillings, and pence.

   British:

   Massachusetts:

   New York:

6. A Virginia blacksmith received a shipment of iron and a bill for £2 3*s* 6*d*. He paid it with 7 Spanish dollars and 2 bits. Use the Rule of Three Direct to find out how much a Spanish dollar was worth in Virginia shillings.

7. With algebra, you can write a Rule of Three Direct problem as an equality of fractions with an unknown term.

   (a) Write #2c on page 1 as an equality of fractions with unknown x; then solve it.

   (b) Write the Rule of Three Direct for three numbers n_1, n_2, n_3, and unknown x as an equality of fractions.

   (c) Show step by step how the Rule of Three Direct comes from your equal-fractions form in part (b).

Name:__________________________________ Date:______________

Activity 2-2

The Rule of Three Inverse

1. Explain why the Rule of Three Direct does not work for this:

   A trucker can make a trip in 4 hours at an average speed of 54 mph. How long will the same trip take at an average speed of 60 mph?

   ____

   ____

The question in #1 is about a different kind of proportionality, called **inverse proportion.** Daboll's *Schoolmaster's Assistant* of 1799 describes its rule like this:

> "The Rule of Three Inverse teaches, by having three numbers given to find a fourth, which shall have the same proportion to the second, as the first has to the third. **If more requires more, or less requires less, the question belongs to the Rule of Three Direct. But if *more* requires *less*, or *less* requires *more*, the question belongs to the Rule of Three Inverse**....
> RULE.---1. State and reduce the terms as in the Rule of Three Direct.
> 2. Multiply the first and second terms together, and divide the product by the third; the quotient will be the answer in the same denomination as the middle term was reduced into."

Notice the order reversal: The 4^{th} number is to the 2^{nd} as the 1^{st} is to the 3^{rd}.

2. (a) Use the Rule of Three Inverse to solve the problem in #1. Show the calculation.

   ____

   (b) Check your result by calculating the distance traveled for each time and rate of speed. At 54 mph: ________________ At 60 mph: __________________

3. If you had been in school in America around 1800, your teacher would likely have asked you to solve these These Rule of Three Inverse problems from Daboll's *Schoolmaster's Assistant*. Show the Rule of Three setup and your calculations.

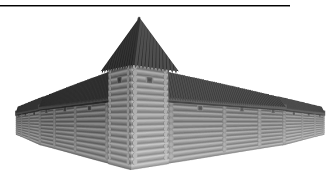
   (a) If 12 men can build a wall in 20 days, how many men can do the same in 8 days?

   ____

   (b) If 30 bushels of grain, at 50 cents per bushel, will pay a debt, how many bushels at 75 cents per bushel will pay the same?

   ____

   (c) Suppose 650 men are in a garrison, and their provisions calculated to last but 2 months, how many men must leave the garrison that the same provisions may be sufficient for those who remain 5 months?



   ____

The Rule of Three Inverse

4. Algebra can help you clarify an inverse proportion problem by writing it as an equality of fractions with an unknown term.

   (a) Write #3a on page 1 as an equality of fractions with unknown x; then solve it.

   ____

   (b) Write the Rule of Three Inverse for three numbers n_1, n_2, n_3, and unknown x as an equality of fractions.

   ____

   (c) Show step by step how the Rule of Three Inverse comes from your equal-fractions form in part (b).

   ____

   ____

   ____

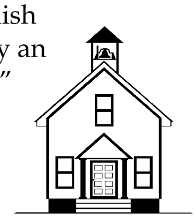
5. Compare your equal-fractions form for the Rule of Three Inverse (#4b above) with your equal-fractions form for the Rule of Three Direct (#7b of sheet 2-1). How can you rewrite one of them to make the relationship between Direct and Inverse clearer? (Think about how to test if two fractions are equal.)

   ____

   ____

Nicolas Pike's 1809 *Arithmetick* says:

> "The principal difficulty that will embarrass the learner will be, to distinguish when the proportion is direct, and when indirect [inverse]. This is done by an attentive consideration of the sense and tenour of the question proposed...."



In other words, understand the problem before you try to solve it!

6. Here are some Rule of Three problems that young Americans studied 200 years ago. In each case, say if the proportion is *direct* or *inverse*.

   (a) If 100£ in 12 months gain 6£ interest, what principal will gain the same in 8 months? ___________

   (b) If a man's wages be 75£ 10*s* a year, what is that a calendar month? ___________

   (c) If 7 yds. of cloth cost 15 dollars and 47 cents, what will 12 yds. cost? ___________

   (d) If 11 men can build a house in 5 months, by working 12 hours per day — in what time will the same number of men do it, when they work only 8 hours per day? ___________

   (e) If 4½ tuns of hay will keep 3 cattle over the winter; how many tuns will it take to keep 25 cattle the same time? ___________

   (f) If a man perform a journey in 5 days, when the day is 12 hours long, how many days will he perform it when the day is but 10 hours long? ___________

   (g) Boarding at 12*s* 6*d* per week, how long will 32£ 10*s* last me? ___________

Name:______________________________ Date:____________

Activity 2-3

False Position

Long before anyone used algebraic symbols, people were solving problems that we write as linear equations. Here's one from an ancient Egyptian papyrus:

A quantity; its fourth is added to it. It becomes 15.

1. Solve this problem by writing an equation to find the unknown quantity. ______________________________

   ____

But the Egyptian scribes didn't know about equations. They solved the problem by "positing" (proposing) an answer that made the arithmetic easy to work with, then using the error to help them find the right answer, like this:

> Suppose the answer is 4 (because it's easy to take a fourth of 4). Take 4 and add its fourth, which is 1. We get $4 + 1 = 5$. But we are supposed to get 15, which is 3 times that. So multiply the proposed answer (4) by 3 to get the correct answer, 12.

This method is called *false position*:

– **Posit (guess at) an answer that you don't expect to be right, but which is easy to work with.**

– **Use your incorrect result to find the number by which you can multiply your guess to get the correct answer.**

2. This problem is like one from the Rhind Papyrus of about 1650 BCE.

   A quantity; its half and its third are added to it. It becomes 44.

   (a) Find this quantity by the method of false position.

   ____

   ____

   (b) Write this problem as an equation; ______________________________

   then solve it. ______________________________________

How do you use the wrong result to find the right answer? By *proportion*:

The wrong result is to the guess as the given result is to the right answer.

Without algebraic equations, that meant using the Rule of Three. Try this approach on the next questions.

3. Use false position and the Rule of Three to solve these three problems from *Welch's Improved American Arithmetic* of 1847. Verify your answers.

   (a) The half, the third, and one fourth of a bag of money made 130 dollars.... I demand how much was in the bag?

   Guess: ____ Calculation: ______________________________

   Rule of Three: ______________________________

   Verify: ______________________________

(b) A schoolmaster, being asked how many scholars he had, said, if I had twice as many, and half as many, and one quarter as many, I should have 264; how many had he?

Guess: ____ Calculation: ____

Rule of Three: ____

Verify: ____

(c) A, B, and C, talking of their ages, B said his age was once and a half the age of A; and C said his age was twice and one tenth of the age of both, and that the sum of their ages was 93; what was the age of each?

Guess: ______ Calculation: ____

Rule of Three: ____

Ages: A ___ B ___ C ___ Verify:______________________________________

4. Write an equation for each of the three problems in #3, and solve them algebraically.

(a) ____

(b) ____

(c) ____

5. Generalize each equation in #4 to a linear equation by using y in place of the given amount in each case.

(a) _________________ (b) _________________ (c) _________________

(d) What are the y-intercepts of these lines? (a) _______ (b) _______ (c) _______

(e) What are their slopes? (a) _______ (b) _______ (c) _______

(f) How are their slopes related to their Rule-of-Three proportions?________________

____

6. Here is a problem from the 1843 edition of Daboll's *Schoolmaster's Assistant*:
A, *B*, and *C* built a house which cost $500, of which *A* paid a certain sum; *B* paid $10 more than *A*, and *C* paid as much as *A* and *B* both. How much did each man pay?

(a) Try solving it by false position:

Guess: ____ Calculation: ____

Rule of Three: ____

Verify: ____

(b) Write an equation for this problem and solve it algebraically.

____

(c) Generalize this equation by using y for the house cost. ______________________

What is the slope of this line? _______ What is its y-intercept? _________

(d) How is this result different from the ones in #3?___________________________

.____

Name:_________________________________ Date:_____________

Activity 2-4

Double False Position

Problem 6 of sheet 2-3 shows that some linear equation problems can't be solved by the false position ("wrong guess") method. In fact, it works only when the equation has zero constant term! If the problem involves a nonzero constant term, can it still be solved without using algebra? Yes, if you make *two* wrong guesses. (Two wrongs *can* make a right!) That's why the process is called **double false position**.

1. To see how double false position works, help with this solution of problem in #6 from sheet 2-3, which said:

   A, *B*, and *C* built a house which cost $500, of which *A* paid a certain sum; *B* paid $10 more than *A*, and *C* paid as much as *A* and *B* both. How much did each man pay?

   (a) First wrong guess: *A* paid $60.
   What would the total cost be in this case? _______ How far off is it? _______

   (b) Second wrong guess: *A* paid $100.
   What would the total cost be in this case? _______ How far off is it? _______

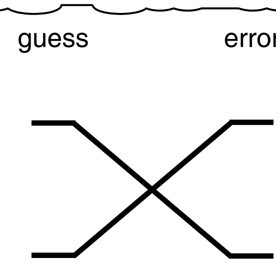
The next part almost seems like a magic trick. In the early 1800s, Nathan Daboll's *Schoolmaster's Assistant* described it like this:

> "Multiply the first position [guess] by the last error, and the last position by the first error.
> If the errors are alike, divide the difference of the products by the difference of the errors, and the quotient will be the answer.
> If the errors are unlike, divide the sum of the products by the sum of the errors, and the quotient will be the answer.
> NOTE.—The errors are said to be alike when they are both too great, or both too small; and unlike, when one is too great, and the other too small."

2. Daboll included a large X diagram, to show how the multiplication was to be done, like the one given here. Fill it in with your information from #1. Then follow his instructions to solve the problem.

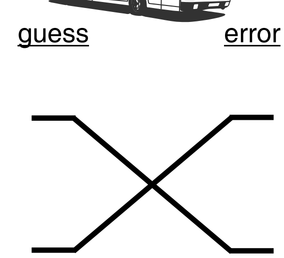
   A paid ______ B paid ______ C paid ______

   Check: ________________________________



3. Solve this problem by double false position.
   A bus with a seating capacity of 39 people starts its run with some people on board.
   - At its first stop, twice as many more people get on and no one gets off.
   - At its second stop, 5 people get off and no one gets on.
   - At its third stop, the number of people doubles, everyone gets a seat, and there is one seat left.

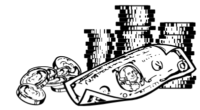
   How many people were on the bus when it started?

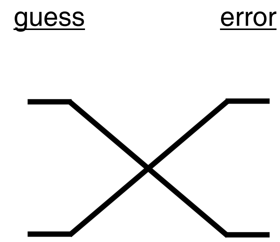


Answer: ______ Check: ____

4. Solve this problem by double false position.
Lucky Lucy invests her first paycheck in the stock of a new company.
   - After six months, her investment is worth \$50 more than three times the amount she started with. She invests that money in a different stock.
   - Four months later, her new investment has tripled in value. She sells it, pays a \$10 commission to the broker, and is left with a total of \$2,003.

   What was the amount of her first paycheck?





Answer: ______ Check: ____

For many centuries before anyone knew how to write, solve, and graph equations, people were using double false position to solve what we call l*inear equation problems* — that is, *problems in which the change in output is proportional to the change in inpu*t. ***But why does it work?*** Maybe the best way to answer that is with some algebraic tools you have studied. We'll help by guiding you through an analysis of another problem.

5. From Daboll's *Schoolmaster's Assistant*: A purse of \$100 is to be divided among four men, *A*, *B*, *C*, and *D*, so that *B* may have \$4 more than *A*, and *C* \$8 more than *B*, and *D* twice as many as *C*; what is each one's share of the money?

   (a) Write this situation as an algebraic equation, using *x* for the amount *A* gets.

   ____ What is its degree? _____

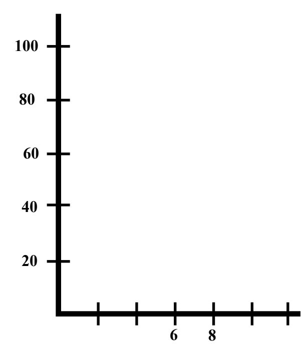
   (b) If you replace \$100 by *y* in the equation, it becomes a linear function of *x*, a straight line determined by any two points on it. That's where the two guesses come in! Solve this problem by double false position, using guesses \$6 and \$8.

   1st Guess: 6. Result: ___ Error ___ 2nd Guess: 8. Result: ___ Error ___

   (c) Part (b) should tell you that (6, 70) and (8, 80) are two points on this line. Graph it on the axes at right.

   (d) The solution is the *x* for which (*x*, 100) is on this line. The slope is constant, so if you compute it by "rise over run" using (6, 70) and (*x*, 100) or by using (8, 80) and (*x*, 100), the result will be the same. Write this equality of fractions:

   ____



   (e) Now cross-multiply: ________________________________

   (f) Solve for *x*: ____

   (g) How are these calculations related to the double false position calculations?

   ____

6. On a separate piece of paper, repeat steps (c) – (g) of #5 for the problem in #4 above. Use your two guesses.

Name:________________________________ Date:______________

Activity 3-1

Completing a Square

1. Many centuries before there were symbols for writing algebra, people were solving problems that we would write as quadratic equations. Perhaps the most famous of these comes from a 9th-century Arabic book by Muhammad ibn Musa al-Khwārizmī:

   > What must be the square which, when increased
   > by ten of its own roots, amounts to thirty-nine?

   Write this as an equation in modern algebraic symbols, using x^2 for "the square." ______________________________

2. Nowadays the general form of a quadratic equation is $ax^2 + bx + c = 0$. But in the 9th century, nobody thought of negatives or zero as numbers, so al-Khwārizmī had to consider six different types of equations for *squares*, *things*, and a *number*. Three of them involve only two terms (e.g., squares equal things), and three involve more than two (e.g, squares and things equal a number). List all six types in words, then in modern notation. Also give a specific example of each one.

In words:	General form:	Example:
squares equal things	$ax^2 = bx$	________________
________________________________	________________	________________
________________________________	________________	________________
squares and things equal a number	________________	________________
________________________________	________________	________________
________________________________	________________	________________

   Which of these six forms is not really a quadratic equation? ________________

   Which form matches the problem in #1? ________________

3. Al-Khwārizmī laid out a recipe for solving the problem in #1 in words, as follows. Fill in the blanks from the information in the problem.

   You halve the number of the roots, which in the present instance yields ______.

   This you multiply by itself; the product is ______________. Add this to thirty-nine;

   the sum is _____________. Now take the root of this, which is _______, and subtract

   from it half the number of the roots, which is ______; the remainder is ______. This

   is the root of the square which you sought for; the square itself is _______.

4. Use al-Khwārizmī's recipe to solve $x^2 + 6x = 40$.
   Check by putting your result back into the equation. x = ___ Check: ______________

5. Apply the Quadratic Formula to $x^2 + nx = A$.

   x = ____

6. Solve $x^2 + nx = A$ by filling in the steps of al-Khwārizmī's recipe.

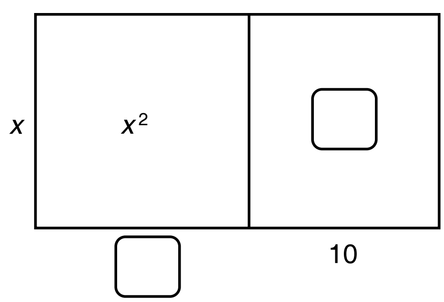
   You halve the number of the roots, which in the present instance yields ___. This you multiply by itself; the product is _____. Add this to ____; the sum is _________. Now take the root of this, which is ___________, and subtract from it half the number of the roots, which is _____; the remainder is ______________. This is the root of the square.

7. Show in detail how your result in #6 is equivalent to the positive-root result in #5.

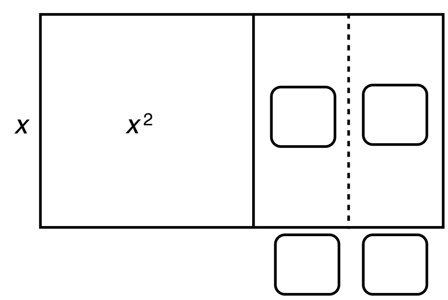
   ____

8. To al-Khwārizmī, the square of a "root" was the area of a square region with that root as its side length. (In modern notation, x^2 is the area of a square with side length x.) He used the following geometric argument to explain why his solution method worked. We have labeled some parts in modern algebraic symbols. The symbols for other parts are missing; fill in the boxes to complete the argument.

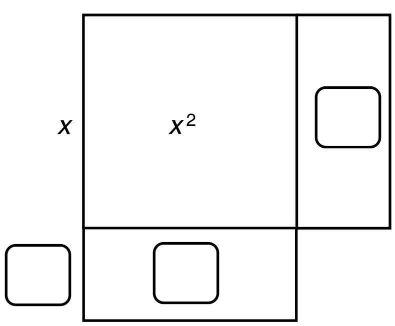
   (a) A "square and 10 roots" is a square of unknown side length and a rectangle with area 10 times that side length:



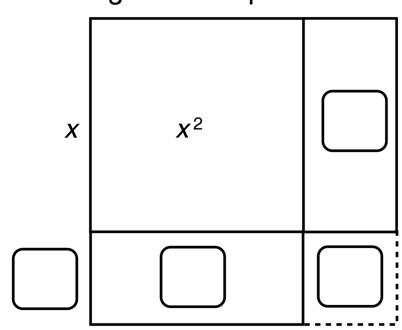
   (b) To "halve the number of roots," bisect the rectangle:



   (c) Now move one half-rectangle to the bottom of the square:



   (d) Complete the (big) square by filling in the missing corner square:



   (e) What is the area of the entire region except for the missing corner square? _____

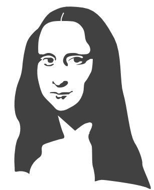
   How do you know? ______________________ What is the area of the completed square? _________ How long is its side? _________ How long is x ? _________

Name:________________________________ Date:______________

Activity 3-2

Algebra Comes of Age

1. The Western world's understanding of algebra began to mature in the 16th century, when people started to write equations in symbols and to accept zero and negatives as numbers. Lots of other important things happened in that century, including most on the following list. For each one that did, write *16th*; for each that did not, write *before* or *after*, depending on when it occurred.



   (a) Martin Luther started the Protestant Reformation. __________

   (b) Leonardo da Vinci painted the *Mona Lisa*. __________

   (c) Gutenberg invented movable-type printing in Germany. __________

   (d) Henry VIII was King of England. __________

   (e) English playwright William Shakespeare was born. __________

   (f) The Pilgrims of the *Mayflower* settled the Plymouth Colony. __________

   (g) Copernicus published his heliocentric theory of the planetary system. __________

Early in the 17th century, Thomas Harriot used these new ideas about numbers and symbols to state a simple principle that made solving polynomial equations much easier than it had been up to then. Harriot was a geographer, a naturalist, and a mathematician.

In 1585 he was sent by Sir Walter Raleigh to help found the first English colony in the New World., on Roanoke Island in an area the British called Virginia. The colony did not survive, but some of Harriot's writings about it still exist. He was its surveyor and its historian.

> **Harriot's Principle:** Move all the terms of the equation to one side of the equal sign, so that the equation has the form [some polynomial] = 0.

2. Rewrite these equations according to Harriot's Principle.

   (a) $7x + 1 = 4x^2 - x + 5$ ________________________

   (b) $2x^3 - 6x + 3 = 4x^2 + 2x - 9$ ________________________

3. By Harriot's Principle, all six of al-Khwārizmī's kinds of quadratic equations can be reduced to special cases of $ax^2 + bx + c = 0$. For each kind, write a specific example in al-Khwārizmī's form; then fill in the appropriate numbers for *a*, *b*, and *c* in the general form given. We'll do the first one for you.

In words:	Example:	$ax^2 + bx + c = 0$
squares equal things	**$3x^2 = 5x$**	$(\mathbf{3})x^2 + (\mathbf{-5})x + (\mathbf{0}) = 0$
squares equal a number	______________	$(\quad)x^2 + (\quad)x + (\quad) = 0$
things equal a number	______________	$(\quad)x^2 + (\quad)x + (\quad) = 0$
squares and things equal a number	______________	$(\quad)x^2 + (\quad)x + (\quad) = 0$
squares equal things and a number	______________	$(\quad)x^2 + (\quad)x + (\quad) = 0$
squares and a number equal things	______________	$(\quad)x^2 + (\quad)x + (\quad) = 0$

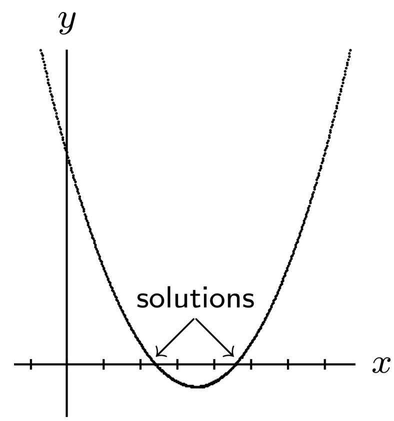
Harriot's Principle was popularized by the famous French mathematician, René Descartes. In 1637, Descartes described a way to picture algebraic expressions by graphing them. (The *Cartesian* coordinate system we use is named for him.) Using Harriot's Principle, the solutions to an equation can be approximated by graphing.

4. To solve $x^2 + 11 = 7x$, for example, first use Harriot's Principle to rewrite it as $x^2 - 7x + 11 = 0$. Then graph the function $y = x^2 - 7x + 11$. The solution to the original equation occurs where the graph crosses the x-axis.

   The picture at right is a graph of this function. Use it to approximate the solutions to $x^2 + 11 = 7x$. Check by substituting into the original equation. See if you can make the difference between the two sides of the equation less than 0.01.

   x = _____; difference: _____

   x = _____; difference: _____



5. Descartes' way of using Harriot's Principle can be applied to polynomial equations of any degree.

   (a) The graph at right is the correct one for solving the equation $x^4 + 2x + 1 = 1.4x^3 + 3x^2$. What function does it represent?

   ____

   (b) How many solutions does this equation have?

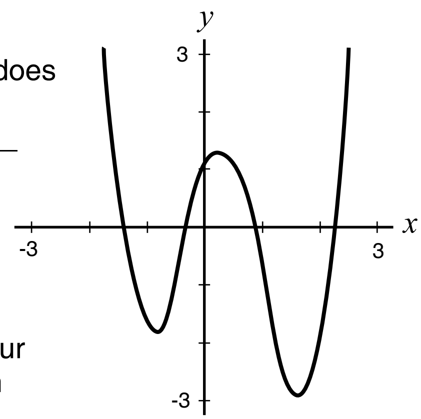
   ____ How do you know?____________________

   ____

   (c) Use the graph to estimate each solution. Check your estimates by putting them into the original equation and finding the difference between the two sides. Try to get an error less than 0.1.

   ____

   ____



6. The graph at right represents $y = 2x^2 - 7x - 4$.

   (a) How is it related to the inequality $2x^2 - 1 < 7x + 3$?

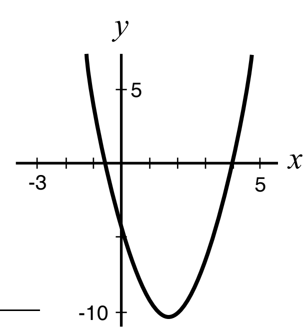
   ____

   (b) Estimate the zeroes of the function to the nearest half-unit; check your answers. ____ and ____.

   (c) For which x-values is the inequality in (a) true? ____________

   Explain. ______________________________________

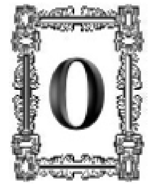
   ____



Name:________________________________ Date:____________

Activity 3-3

Using Zero



A special property of zero in our number system makes Harriot's Principle (on Sheet 3-2) a powerful tool for solving polynomial equations.

1. Can you find two nonzero numbers whose product is 0?
   If so, do it. If not, why do you think it can't be done? ________________

   ____

   ____

2. What can you say about numbers a and b in each case?

   (a) $3a = 3b$ ________ (b) $47a = 47b$ ________ (c) $0a = 0b$ ________

3. Use your answer to #1 to justify your answer to #2(a). ______________________

   ____

Question 1 points to an important fact:

If the product of two numbers is 0, then at least one of them must be 0.

Mathematicians refer to this fact by saying that our number system has *no zero-divisors*. That's <u>*not the same*</u> as saying,

"If at least one of two numbers is 0, then their product is 0."

In fact, it's the *converse* of that statement, which means that the hypothesis (the *if* part) and the conclusion (the *then* part) are reversed.

4. Rewrite equations (a), (b), and (c) according to Harriot's Principle. Then use the fact that there are no zero-divisors to rewrite them so that the leading coefficient (the coefficient of the highest-power term) is 1.

   (a) $7x + 1 = 4x^2 - x + 5$ ________________ ________________

   (b) $2x^3 - 6x + 3 = 4x^2 - 9$ ________________ ________________

   (c) $3x + 8 = 5x^2 + x + 1$ ________________ ________________

   (d) Explain how "no zero-divisors" lets you do this. ____________________

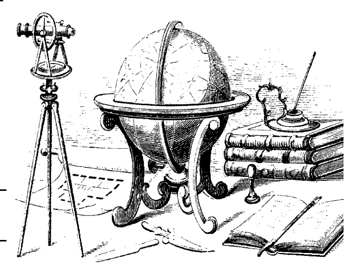
   ____

5. By the end of the 16th century, European mathematicians knew a lot about factoring polynomials. For example, they knew that $x^2 - 4x + 3 = (x - 3)(x - 1)$. When Harriot's Principle told them that $x^2 + 3 = 4x$ was the same as $x^2 - 4x + 3 = 0$, they could immediately solve that equation. How? Do it, and check.

   ____

   ____

   ____



Using Zero

6. The idea behind #5 is that, if you can break a quadratic polynomial into two linear factors, then solving the two linear equations gives you the solutions to the quadratic. Use that method to solve these equations. Check your answers by substituting them back into the original equation.

   (a) $x^2 + 3x - 4 = 0$ Factored form: ______________________ $x =$ ______, ______

   (b) $4x^2 - 36 = 0$ Factored form: ______________________ $x =$ ______, ______

   (c) $x^2 - 11x = 0$ Factored form: ______________________ $x =$ ______, ______

   (d) $2x^2 + 20x = -50$ Factored form: ______________________ $x =$ ______, ______

   (e) $x^2 + 4 = 5x$ Factored form: ______________________ $x =$ ______, ______

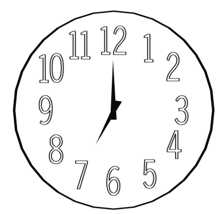
7. Sometimes higher-degree equations can be factored into quadratics that can then be broken into linear factors, as you did in #6. Even if they can't be broken down completely, you may be able to get some solutions. See if you can do these.

   (a) $x^3 + 2x^2 + x = 0$ Factored form: _________________________ $x =$ ____________

   (b) $x^5 - 25x^3 = 0$ Factored form: _________________________ $x =$ ____________

   (c) $x^4 - 16 = 0$ Factored form: _________________________ $x =$ ____________

Cancellation and no zero-divisors are such familiar properties that it is hard to imagine any worthwhile arithmetic system without them. But there are some. The following activities look at one of them, sometimes called "clock arithmetic." It is one of the modular arithmetic systems developed by the German mathematician Carl Friedrich Gauss at the end of the 18th century.



8. This clock arithmetic system uses only the numbers 0, 1, 2, 3, ..., 11. Addition and multiplication are defined by doing the usual addition and multiplication, dividing by 12, and taking the remainder. For instance, 6 + 10 = 4 and 5 x 7 = 11.

   (a) Calculate: 5 + 9 = ___; 7 + 8 = ___; 11 + 1 = ___; 3 + 4 = ___; 10 + 11 = ___

   (b) Calculate: 4 x 5 = ___; 7 x 3 = ___; 6 x 4 = ___; 3 x 1 = ___; 3 x 5 = ___

   (c) What do your last two answers to (b) tell you about cancellation? Why?

   ____

9. In our usual system, linear equations have at most one solution and quadratics have at most two. Not so in the system of integers modulo 12. Find all solutions of these equations in that system. Check your answers.

   (a) $2x - 4 = 0$; $x =$ _______________ (b) $x^2 - 1 = 0$; $x =$ _______________

   (c) $2x^2 - 3x = 0$; $x =$ _______________ (d) $3x^2 + 6x = 9$; $x =$ _______________

10. Are $2x - 6 = 0$ and $x - 3 = 0$ equivalent equations in the integers modulo 12? _____

    Justify: ____

Name:__________________________________ Date:______________

Activity 3-4

A Method That Always Works

It is not always easy to break a quadratic polynomial into linear factors. The ones on sheet 3-3 were carefully chosen to illustrate Harriot's Principle at work. But most quadratics are not so nice. al-Khwārizmī's approach (sheet 3-1) worked well, but he didn't have a notation system to turn his examples into a formula that would always work.

1. Our current algebraic notation system emerged in the late 17th century, as European mathematicians began to accept Descartes' practice of using letters at the end of the alphabet for variables and letters at the beginning for constants.

   (a) Write three specific examples of the equation $ax^2 + bx - c = cy + a$.

   __________________; __________________; __________________

   (b) Write the general form of $5x - 3x^2 + 6 = 2y^3 + 7y$. ______________________

   (c) $5x^2 + 8x + 9$ fits one of the forms $ax^2 + bx + c$ and $ax^2 + bx + a$, but not the other.

   Explain. ____

   Write a polynomial that fits both forms. ______________________

By the end of the 17th century, mathematicians had become fairly comfortable with negative numbers and algebraic notation. Work through the following questions to see how these tools can be used to turn al-Khwārizmī's approach into a formula for solving any quadratic equation!

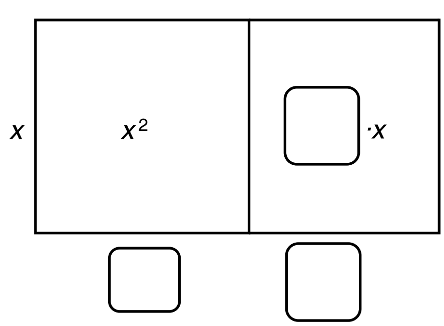
2. Rewrite the general quadratic equation $ax^2 + bx + c = 0$ as "a square and some roots equals some number": $x^2 +$ _____ $x =$ _____.

   To use al-Khwārizmī's method of completing the square as in Sheet 3-1, you need to think of your answers as positive numbers. Why is that?

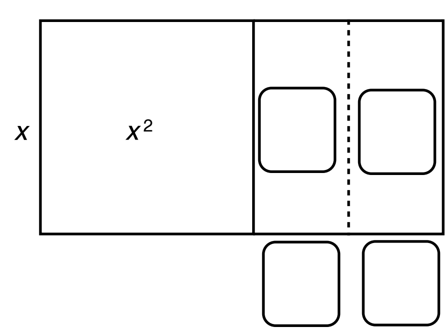
   ____

3. Here are the diagrams for al-Khwārizmī's argument. Some parts of them are labeled. Use your answers to #2 to fill in the missing labels for the other parts.

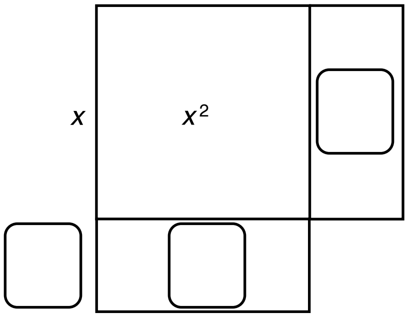
   (a) A "square and some roots" is a square of unknown side length and a rectangle with area some number of times that side length:



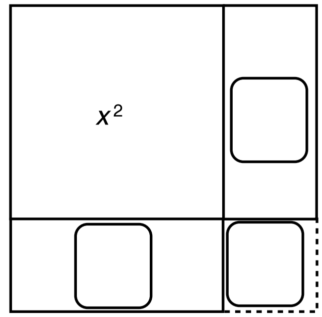
   (b) Now divide into two halves the area representing the "number of roots" by bisecting the rectangle:



(c) Now move one of the rectangular halves to the bottom of the square:



(d) Complete the big square by filling in the missing corner square:



(e) Express the combined area of the square and rectangle in (a) in two different ways. __________or ________ How is that related to the area of the region in (d) without the missing corner square? _______________

What is the area of the completed square? ______________

How long is its side? _________ How long is x ? _____________________

(f) Show step by step how your answer for x can be rewritten as $\dfrac{-b+\sqrt{b^2-4ac}}{2a}$.

____

____

By the early 18th century, it was accepted that a positive number has *two* square roots, even though the negative one was not a side length of a square. This means that the expression in #3(f) actually provides a formula for two solutions of $ax^2 + bx + c = 0$:

$$x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$$

This is called the **Quadratic Formula**, and it works for any quadratic equation.

4. To verify that the Quadratic Formula always works, you can put these two expressions in place of x in $ax^2 + bx + c$ and simplify the algebra to see that you actually get 0 in each case. We suggest this as a challenge to your care, patience, and ingenuity. (Get a piece of scrap paper and try it! It's not so bad, after all.)

5. Mathematicians of the early 18th century were bothered by the possibility that $b^2 - 4ac$ might be negative. Why do you think it concerned them? ______________________

____

6. (a) Use the Quadratic Formula to solve $3x^2 + 8x - 17 = 0$. Check your answer.

x = ____

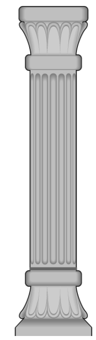
(b) Make up a quadratic equation ____________________, then use the Quadratic Formula to solve it. x = ___________ or ___________. Check your answer.

Name:______________________________ Date:____________

Activity 3-5

Quadratics in Earlier Times

Long before algebra as we know it, people were working with quadratic equations. Here are a few instances from the centuries before the tools of algebra were available. (You will need some scrap paper as you work through them.)



1. In the 2nd century, The Greek mathematician Diophantus posed this problem:

   To find two numbers such that their sum and the sum of their squares are given numbers.

   (a) To illustrate his solution, Diophantus used 20 as the sum and 208 as the sum of the squares. Use these values to write the problem as a quadratic equation *in one unknown*. Then solve it (any way you want).

   Equation: ______________________ x = _____ or _____

   (b) Write a quadratic equation for the general form of this problem, using a for the sum and b for the sum of the squares.

   Equation: ______________________________

   (c) Use the Quadratic Formula to solve the general problem in terms of a and b.

   x = ______________ or ______________

   (d) Diophantus said that to solve this kind of problem it is necessary that "double the sum of the squares exceed the square of the sum by a square." Why do you think he said this? (Look at your answer to part (c).)

   ____

   Is it really necessary? Explain. ____________________________________

   ____

2. Abū Kāmil (ca. 850–930) lived (probably) in Egypt, not too long after al-Khwārizmī. In his algebra book, he says that he can solve equations that are not of one of al-Khwārizmī's six standard types (see Sheet 3-1). Here is one of his problems:

   One says that 10 is divided into two parts, each of which is divided by the other, and when each of the quotients is multiplied by itself, and the smaller [product] is subtracted from the larger, then there remains 2.

   (a) We can write "10 is divided into two parts" as $v + w = 10$, and assume that $v > w$. Using that notation, write an equation for the rest of the problem.

   ____

   (b) Abū Kāmil thought of the larger quotient as a single unknown, which we'll call x. Rewrite your equation of part (a) in terms of x, as a polynomial equation.

   ____

(c) Solve your equation in (b) as a quadratic in the unknown x^2. x^2 = ______________
(Abū Kāmil would have accepted only the positive solution here.)

(d) Now rewrite x in terms of w (only) to get a quadratic equation in w, and solve it.

x = __________ Equation: ______________________ w = ____________ (positive)

(e) If you have a calculator, evaluate w to one decimal place: ____. Putting that value into the original equation, how close do you come to 2? Error: ______

3. As early as the 17th century BCE, Babylonian scribes were doing quadratic problems. To decipher what they wrote, you need to understand their base-sixty place value system. Explaining their wedge-shaped symbols here would take too long. Just know that they had symbols for the numbers 1 to 59, which we'll write in our usual way.
    In base sixty, each place value is a power of 60. We'll separate the places by commas, and use a semicolon to set off the integral part from the fractional part. For instance, 2,7;15,30 means $2 \cdot 60 + 7 \cdot 1 + 15/60 + 30/60^2 = 127\ 31/120$.

(a) Write each of these numbers in our usual notation:

4;45 = ___________ 3,20;30 = ___________ 1,2;10,20 = ___________

(b) Write each of these numbers in base sixty:

25¼ = ___________ 63½ = ___________ 245¾ = ___________

Here is an example of a solved problem from a Babylonian scribe's tablet.

I have added up the area and the side of my square: 0;45. You write down 1, the coefficient. You break off half of 1: 0;30 and 0;30 you multiply: 0;15. You add 0;15 to 0;45: 1. This is the square of 1. From 1 you subtract 0;30, which you multiplied. 0;30 is the side of the square.

(c) What is the question? That is, what problem is being solved here?

____

____

(d) Write the problem as a quadratic equation and solve it by the Quadratic Formula.

Equation: _____________________________ Solution: ___________________

(e) Use the Quadratic Formula to show how the solution of an equation of the form $x^2 + bx - c = 0$ can be written as an expression involving only $b/2$ and c.

____

(f) Explain the steps of the Babylonian solution in terms of (e). We'll start you out.

You write down 1, the coefficient. This is b. You break off half of 1:This is _______.

0;30 and 0;30 you multiply: 0;15. ________________ You add 0;15 to 0;45: 1.______

___________________ This is the square of 1. ______________ From 1 you subtract

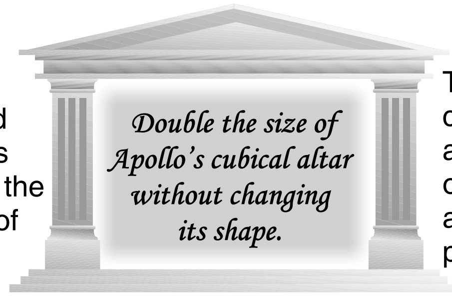
0;30...__________________________________ 0;30 = _____ is the side of the square.

Name:________________________________ Date:______________

Activity 4-1

Boxes, Lines, & Angles

1. In 430 BCE, a plague was raging in Athens. Legend tells us that the Athenians appealed to the oracle of the god Apollo on the island of Delos. The oracle said,



They built a new altar, a cube with each edge twice as long as the edges of the old one. Their mistake angered Apollo, and the plague got even worse!

(a) What is an oracle? ______________________________

(b) What was the Athenians' mistake? ______________________________

____

(c) What did the oracle really want them to do? ______________________________

____

(d) Using x for the unknown edge length of the new altar, write the oracle's problem (the Delian Problem) as an equation; then solve it.

____

(e) Give at least two reasons why this problem would be very hard for the Greeks.

(1) ______________________________

(2) ______________________________

2. In the 5th century BCE, Hippocrates of Chios showed how the Delian Problem was related to finding two lengths to form a double mean proportional. In modern symbols, if a is the original edge length, and if you can find two lengths x and y such that $\frac{a}{x} = \frac{x}{y} = \frac{y}{2a}$, then x is the edge length of the doubled altar.

(a) To see how a double mean proportion works, find numbers x and y such that $\frac{8}{x} = \frac{x}{y} = \frac{y}{27}$. Check by substitution.______________________________

____

____

(b) Show how Hippocrates' double mean proportional can be rewritten to form a cubic equation in one unknown, x. ______________________________

____

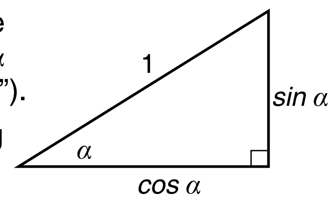
Is it the same as your equation for 1(d)? _______ ______________________________

____

The Greeks eventually solved the Delian Problem geometrically, using conic sections, but they never found a compass-and-straightedge solution. It was another 2000 years before mathematicians proved — algebraically — that such a construction was impossible.

Another ancient problem that cannot be solved by compass and straightedge alone is finding a way to divide any given angle into three equal parts (i.e., to *trisect* it). By the 10th century, Indian scholars knew a lot about how angles relate to ratios of line segments, a field we now call *trigonometry* (which means "triangle measurement"). This paved the way for using algebra to solve this and other problems of angle measurement. Here, in modern language, are some basic trigonometric ideas that the Indians knew.

3. If an angle α is at one vertex of a right triangle with hypotenuse 1 unit long, then the length of the side opposite α is called $\sin\alpha$ ("sine") and the length of the other side is called $\cos\alpha$ ("cosine").



(a) Use the Pythagorean Theorem to write an equation relating $\sin\alpha$ and $\cos\alpha$.

____

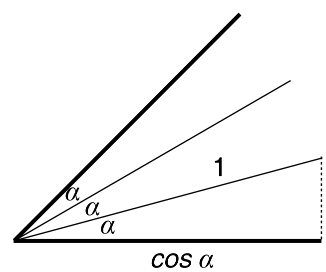
(b) The Indians knew that $\cos(\alpha+\beta) = \cos\alpha \cdot \cos\beta - \sin\alpha \cdot \sin\beta$. Use this to write a formula for $\cos 2\alpha$ in terms of $\cos\alpha$ and $\sin\alpha$.

$\cos 2\alpha =$ ____________________________________

(c) They also knew that $\sin(\alpha+\beta) = \sin\alpha \cdot \cos\beta + \cos\alpha \cdot \sin\beta$. Use this to write a formula for $\sin 2\alpha$ in terms of $\cos\alpha$ and $\sin\alpha$.

$\sin 2\alpha =$ ____________________________________

4. The Indians knew that cosine of an acute angle determines the angle. This means that you can trisect a given angle θ if you can find $\cos\theta/3$.



Assume that θ is a known angle, and hence its cosine is a known length. Let $\alpha = \theta/3$, and let $x = \cos\alpha$. Use your answers to #3 to help you write $\cos\theta$ as a cubic polynomial in x. (You'll need some scrap paper.)

$\cos\theta = \cos 3\alpha =$ ____

____

5. Indian and Arab mathematicians thought of sines and cosines as line segments.

(a) Explain how knowing the cosine of an acute angle determines the angle.

____

____

____

(b) If the length of $\cos\theta$ is 0.82, what is the length of $\sin\theta$ (to two decimal places)?

Show work. ____

(c) Use the information of part (b) to calculate the slope measure of θ to one decimal place. ______________

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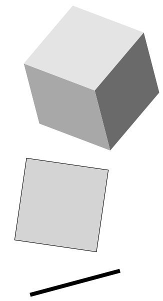
Activity 4-2

From Shapes to Numbers



Much of 9th- and 10th-century mathematics was based on the geometry of shape and measurement. When the Indian and Arab mathematicians of that era talked about cubes, they were thinking of boxes with square corners and equal edges. Because numbers represented lengths or areas or volumes, negatives and zero were not allowed. Al-Khwārizmī's 9th-century book that was the beginning of algebra described equations entirely in words, without using negative numbers or zero.

1. Al-Khwārizmī might have written $x^3 = 7x + 6$ something like this:

"A cube is six more than seven of its roots."

"Root" meant the edge of the cube, so this said that the volume of the cube was seven times its edge length plus 6.

(a) Write each of these equations in modern symbols.

(i) A cube and three of its roots is fourteen. ______________________

(ii) Twice a cube is ten more than three of its roots. ______________________

(iii) A cube and five is twenty-six of its roots. ______________________

(b) These four cubic equations were carefully chosen to have easy solutions. Can you find them?

sample _____ (i) _____ (ii) _____ (iii) _____

2. In the 11th century, some Arab scholars used Al-Khwārizmī's methods to study cubic equations. The most famous of these was 'Umar al-Khāyammī, an Iranian mathematician better known in Western culture as Omar Khayyam.

(a) Omar Khayyam was also a philosopher, astronomer, and poet.

What is his most famous written work? _______________

What kind of work is it? ___________________________

Without negative numbers and zero, Khayyam had many different *types* of cubic equations. For example, $x^3 = ax + b$ (a cube equals roots and a number) and $x^3 + ax = b$ (a cube and roots equal a number) were different types. In fact, he needed to solve 19 types, starting with the number of terms in each one.

(b) Why do you think he ignored these 5 types?

$x^3 = ax^2$, $x^3 = ax$, $x^3 + ax^2 = bx$, $x^3 + ax = bx^2$, $x^3 = ax^2 + bx$

____

(c) Find all roots of each of these equations. Which would Khayyam accept as roots?

$x^3 = 4x$ roots: _________________ Khayyam's roots: ________

$x^3 + 3x^2 = 4x$ roots: _________________ Khayyam's roots: ________

$x^3 + 2x = 3x^2$ roots: _________________ Khayyam's roots: ________

3. (a) Khayyam considered 6 other types of three-term equations. One of them was $x^3 + a = bx$ (a cube and a number equal roots). What are the others?

   In symbols: ______ In words:

   ______ A cube and squares equal a number. ______

   ______ ______

   ______ ______

   ______ ______

   ______ ______

   (b) Khayyam's forms only referred to "a cube" (x^3). Why do you think he didn't bother to consider other coefficients for this term?

   ______

   (c) Rewrite each of these equations to fit one of the forms in part (a).

   $3x^3 = 2x^2 + 12$ ______

   $5x^3 + 8x = 3$ ______

   $(¾)x^3 + 12x^2 = 5$ ______

4. Besides the six types of three-term cubic equations listed in #3, mathematicians of Omar Khayyam's time had to consider seven different types of four-term cubic equations. One of them is $x^3 + ax^2 = bx + c$. Can you find the other six?

   ______ ______ ______

   ______ ______ ______

Omar Khayyam found a *geometric* solution — a line segment whose length satisfied the equation — for each of his 14 types of cubics. That was impressive, but even he noted that they didn't provide *numerical* solutions. That had to wait 500 more years, until the notation of algebra was good enough and a concept of *number* was abstract enough to admit that zero and negatives were numbers. Equation notation finally settled down in the 17th century, after the work of René Descartes and Thomas Harriot.

5. By Harriot's Principle, all of Khayyam's many types of cubic equations can be written in a single standard form: $x^3 + ax^2 + bx + c = 0$, where a, b, and c are real numbers. Write each of the following equations in that form by filling in the proper coefficients.

   (a) $x^3 + 6x = 7 \quad \Leftrightarrow \quad x^3 + (\_\_\_)x^2 + (\_\_\_)x + (\_\_\_) = 0$

   (b) $x^3 = 4x^2 + 5x + 1 \quad \Leftrightarrow \quad x^3 + (\_\_\_)x^2 + (\_\_\_)x + (\_\_\_) = 0$

   (c) $x^3 + 3 = 2x^2 \quad \Leftrightarrow \quad x^3 + (\_\_\_)x^2 + (\_\_\_)x + (\_\_\_) = 0$

   (d) $x^3 + x = 8x^2 + 9 \quad \Leftrightarrow \quad x^3 + (\_\_\_)x^2 + (\_\_\_)x + (\_\_\_) = 0$

   (e) $x^3 = 19x \quad \Leftrightarrow \quad x^3 + (\_\_\_)x^2 + (\_\_\_)x + (\_\_\_) = 0$

   (f) $6x^3 = 8x + 9 \quad \Leftrightarrow \quad x^3 + (\_\_\_)x^2 + (\_\_\_)x + (\_\_\_) = 0$

   (g) $x^3 + 5x^2 + 11 = 24x \quad \Leftrightarrow \quad x^3 + (\_\_\_)x^2 + (\_\_\_)x + (\_\_\_) = 0$

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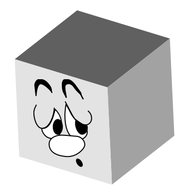
Activity 4-3

The Depressed Cubic

There wasn't much progress on finding a general method for solving cubic equations until the 15^{th} century in Italy. At that time, a successful mathematician had to engage in one-on-one public contests of problem solving. These "mathematical duels" provided fame and patronage to the winners, and professional disgrace to the losers.

A famous contest took place in 1535. Scipione del Ferro discovered how to solve cubic equations that had no square term. He kept his powerful "weapon" secret until he was dying, when he passed it on to his student, Antonio Fiore. Unwisely, Fiore used it to challenge a prominent mathematician known as Tartaglia, with 30 problems of this kind. Tartaglia worked frantically throughout the time allowed for the contest and finally found a method. He solved all of Fiore's problems, while Fiore did very poorly on the 30 varied problems Tartaglia had posed in return.



1. This contest became famous because of the method on which it turned. Del Ferro and Tartaglia had found a way to solve equations of the form "**a cube and things equal a number.**" These are called *depressed cubics*. (No, they are not unhappy boxes; they are cubic equations without a square term.)

   (a) Write "a cube and things equal a number" in modern notation. ________________

   (b) Give two specific examples of depressed cubics.

   ________________________ and ________________________

   (c) Because they did not use negative numbers, 15^{th}-century mathematicians would have considered two other forms of depressed cubics, besides the one given here. Write those other two forms in words and give a specific example of each.

   ____ ____________________

   ____ ____________________

The way to solve depressed cubics was kept secret until 1585, when it appeared in Girolamo Cardano's *Ars Magna* ("The Great Art"). Cardano's devious role in uncovering Tartaglia's "secret weapon" is a fascinating story of intrigue and deceit. You can read all about it in Chapter 6 of William Dunham's *Journey Through Genius* (Wiley, 1990).

2. Cardano described the method using the example "**a cube and 6 things equals 20.**" Here are his steps; you supply the calculations for his example, in modern notation.

   (a) To solve: A cube and 6 things equals 20. To solve for x : __________________

   (b) Cube one-third the coefficient of x; add to it the square of one-half the constant; and take the square root of the whole. ______________

   (c) Repeat this, and to one of the two you add ½ of the constant and from the other you subtract ½ the same.________________________

   (d) Subtract the cube root of the first from the cube root of the second; the remainder is the value of x. __________________________

   (e) Evaluate your answer to (d) with a calculator, and check the result by substitution into the equation. __________________________

Were you surprised by the result when you put that ugly formula intoyour calculator? Sometimes solving one problem brings up another one. Cardano's method leads to the question of how to take a messy formula and recognize that it can be simplified.

3. Try this one on your own. Solve $x^3 + 12x = 36$. Calculate your answer to two decimal places, and check by substitution.

 ____

 _____________ and _____________

 $x =$ ________________________________

 Check: ___________________________________

4. A general formula for solving any depressed cubic can be found by applying Cardano's method to an equation with literal coefficients, say $x^3 + cx = k$. Do it.

 ____

 ____________________ and ____________________

 $\boxed{x = \qquad\qquad\qquad\qquad}$

 Keep this formula. You'll need it for Sheets 4-4 and 4-5.

5. Use Cardano's formula to find a solution for each equation.

 (a) $x^3 + 9x = 26$ ____

 Check by substitution: _______________________________________

 (b) $x^3 + 3x = 4$ ____

 Is this a whole number? ______ How do you know? ___________________________

 ____

6. Because 15th- and 16th-century mathematicians did not use negative numbers, Cardano had to deal with two other kinds of depressed cubics as separate problems. Today we can just use negatives to rearrange those equations and then apply his original formula. Rearrange and solve these equations and check by substitution.

 (a) $x^3 = 21x + 90$ Rearranged: ____________________

 Solution: ____

 Check: _______________________

 (b) $x^3 + 40 = 6x$ Rearranged: ____________________

 Solution: ____

 Check: _______________________

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Activity 4-4

The General Cubic

What about solving cubic equations that are not depressed — ones with no x^2 term? The key is to turn this problem into one you already know how to solve. In this case, that meant finding a way to turn any cubic equation into a depressed cubic. When Cardano found a way to do this, he published Tartaglia's secret method for depressed cubics without permission because he wanted to present his own generalization to the world!

As before, Cardano's work was made much more tedious by the need to use only positive coefficients and the lack of good notation in the 16th century. Modern notation and negative numbers will make it easier for you to follow his ingenious argument without getting bogged down in all the special cases he had to consider.

1. Cardano's touch of genius was his choice of just the right substitution. By replacing the original variable with a related expression, he got a cubic without a squared term, which could be solved by Tartaglia's method. Here is his argument in modern notation, starting with the general equation $\boldsymbol{x^3 + ax^2 + bx + c = 0}$.

   (a) The key: Let $y = x + a/3$, so $x = y - a/3$.

   Expand: $(y - a/3)^2 =$ ____________________

   Expand: $(y - a/3)^3 =$ __________________________

   (b) Substitute these results into the general equation and multiply it out, to get a cubic equation in the variable y:

   (__________________) + a(________________) + b(________) + $c = 0$

   y^3 + (___)y^2 + (______________)y + (__________________) = 0

   (c) This a depressed cubic equation because ______________________________.

2. Cardano did not use such general notation, of course, because it wasn't available at that time. Instead, he used typical examples to present his ideas. We'll help you do that here. Work through the method of #1 using $\boldsymbol{x^3 + 6x^2 + 10x + 8 = 0}$.

   (a) $y =$ _________, so $x =$ __________.

   Expand: (______)^2 = ____________________

   Expand: (______)^3 = __________________________

   (b) Substitute these results into the general equation and multiply it out, to get a depressed cubic equation in the variable y:

   (__________________) + __(_______________) + __(________) + __ = 0

   Depressed cubic in y : ____________________________

   (c) Solve this equation by the Tartaglia/Cardano formula, which you found in #4 of Sheet 4-3; evaluate it with a calculator.

   $y =$ ____ , so $y =$ ______

(d) Now finish the job. Find x : __________. Substitute to check that this is a correct solution for the original equation.

Check:____

3. What if the equation's leading coefficient is not 1? How can you use Cardano's approach for an equation like $5x^3 + 8x^2 + 4x + 7 = 0$? (Don't actually solve this one.)

____

4. Cardano also explained some "tricks" for solving some (but not all) troublesome equations. One of the equations he solved with a trick was $x^3 + 18 = 15x$.

(a) The solution he found was $x = 3$; check it: ______________________________

(b) Try to solve this equation using Cardano's formula.

____

(c) From Cardano's viewpoint, what's "wrong" with the result in (b)?

____

(d) Find the two other solutions for this equation. (Begin by factoring out $x - 3$.)

____

____

____

5. Tartaglia and Cardano lived during the height of the Italian Renaissance. It was a time of great art and literature and of new directions in music and science. Here are some famous Italians of that time. Look up and list the life span of each one, the field in which he is best known, and one famous work, as follows:

Cardano	(1501–1576)	Mathematics	*Ars Magna* (book)
Michelangelo	___________	___________	___________
Palestrina	___________	___________	___________
Raphael	___________	___________	___________
Galileo	___________	___________	___________

6. Piero della Francesca, a famous 15th-century painter, also wrote mathematical works. In one of them, he gave this recipe for solving 4th-degree equations:

> When things and squares and cubes and squares of squares are equal to numbers, one should divide the number of things by the number of cubes, square the result and add to the number. Then the thing will be equal to the square root of the square root of the sum minus the root of the result of dividing things by cubes.

(a) Write this recipe in symbols as a formula for solving $x^4 + ax^3 + bx^2 + cx = d$.

$x =$ ______________________________________

(b) What about this formula should make you suspect that it won't always work?

____

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Activity 4-5

Impossible, Imaginary, Useful

1. (a) 5^2 = _____ (b) $(-5)^2$ = _____ (c) $(\sqrt{5}\,)^2$ = _____

   Assume that *k* is a real number. Circle the correct answer in each case:

   (d) If *k* is positive, then k^2 must be ____ (positive, negative, can't tell).

   (e) If *k* is negative, then k^2 must be ____ (positive, negative, can't tell).

   (f) If If k^2 is positive, then *k* must be ____ (positive, negative, can't tell).

2. Mathematicians of the 16th century thought of algebraic equations as ways to solve concrete problems about length, distance, weight, etc. In his *Ars Magna* of 1585, Cardano considers the problem of finding two numbers with sum 10 and product 40.

   (a) He already knew that no such numbers actually exist. How? (Remember that graphing equations was unknown in the 16th century. Think about rectangles.)

   ____

   ____

   (b) Cardano used the Quadratic Formula to find an algebraic solution. What was it?

   ____

   (c) Verify that your solution satisfies the conditions of the problem.

   ____

   ____

3. Cardano didn't think of his algebraic answers as real solutions because there was no place for them in the real numbers. This view seemed to be confirmed by Descartes' graphs of algebraic expressions in the first half of the next century.

   (a) When you solve an equation of the form $ax^2 + bx + c = 0$ by graphing, where do the solution points appear? _______________________________________

   (b) When you try to solve the equation for Cardano's problem in #2 by graphing, what goes wrong? ____

Whenever the process of solving a quadratic equation produced the square root of a negative number, mathematicians of the 16th and 17th centuries called them **imaginary** or fictitious and said that the equation had no **real** solution. Descartes' use of these terms led to their common acceptance by the end of the 17th century.

4. Cardano's formula for solving cubic equations often produces square roots of negatives. Do you think this means that those equations have no real solutions?

   Why or why not? (Think of graphs.) _______________________________________

   ____

5. The idea that imaginary numbers meant that there were no real solutions *was wrong for cubics*. Here is an example.

   (a) Use Cardano's formula (from Sheet 4-3) to solve $x^3 = 15x + 4$.

   $x =$ ____

   (b) You got something imaginary, right? But $x = 4$ is a *real* solution!

   Check it: ____

   (c) Well, a cubic equation has up to three different solutions. To see if the others are imaginary, rewrite $x^3 = 15x + 4$ as $x^3 - 15x - 4 = 0$ and factor out $x - 4$.

   (You'll need some scrap paper.) $x^3 - 15x - 4 = (x - 4)\cdot($__________________$)$

   (d) Now use the Quadratic Formula on the other factor to get the other two solutions:

   $x =$ ____________ or ____________ . Are they real or imaginary? ____________

   (e) If you have a graphing calculator, graph together the original cubic and the quadratic you found. How does this show that they have two solutions the same?

   ____

6. Cardano didn't know how to handle equations like the one in #5. But in the 1560s, Rafael Bombelli proposed some rules for working with these "imaginary" things (still not regarded as actual numbers). In modern notation, they worked like this:

   $(\sqrt{-a})(\sqrt{-a}) = -a \qquad (-\sqrt{-a})(-\sqrt{-a}) = -a \qquad (\sqrt{-a})(-\sqrt{-a}) = a$

   (a) Use Bombelli's rules to help you calculate these:

   $(2 + \sqrt{-1})^3 =$ __________________ $(2 - \sqrt{-1})^3 =$ __________________

   (b) Use your answers from (a) to show how Cardano's "imaginary" solution of the equation in #5(a) actually equals 4. (Cardano was right, after all!)

   ____

   ____

7. Square roots of negatives were neither positive nor negative, so there was no place to put them on the (real) number line. That made them hard to accept, but they were useful, so people began to work with them. It became customary write the square root of any negative as a real number times $\sqrt{-1}$; for instance, $\sqrt{-9} = 3\sqrt{-1}$.

   (a) Write each of these imaginaries in the form of the example.

   $\sqrt{-25} =$ __________ $\sqrt{-100} =$ __________ $\sqrt{-5} =$ __________ $\sqrt{-12} =$ __________

   (b) By the early 1800s, Gauss and other German mathematicians had begun using i as a shorthand for $\sqrt{-1}$. Write each imaginary of part (a) in this form.

   __________ __________ __________ __________

Thus, the imaginaries form a copy of the real line with an i on each number. You can think of that line as perpendicular to the real line, as Paris bookseller J. R. Argand did in 1806. Then the plane represents all things of the form $a + bi$, where a and b are real numbers, a picture of the **complex number system**. But that's a story for another time.