Overhang*

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1. INTRODUCTION. How far off the edge of the table can we reach by stacking n identical, homogeneous, frictionless blocks of length 1? A classical solution achieves an overhang asymptotic to $\frac{1}{2}\ln n$. This solution is widely believed to be optimal. We show, however, that it is exponentially far from optimality by constructing simple n-block stacks that achieve an overhang of $cn^{1/3}$, for some constant $c > 0$.

The problem of stacking a set of objects, such as bricks, books, or cards, on a tabletop to maximize the overhang is an attractive problem with a long history. J. G. Coffin [**2**] posed the problem in the "Problems and Solutions" section of this MONTHLY, but no solution was given there. The problem recurred from time to time over subsequent years, e.g., [**18**], [**19**], [**12**], [**4**]. Either deliberately or inadvertently, these authors all seem to have introduced the further restriction that there can be at most one object resting on top of another. Under this restriction, the *harmonic stacks*, described below, are easily seen to be optimal.

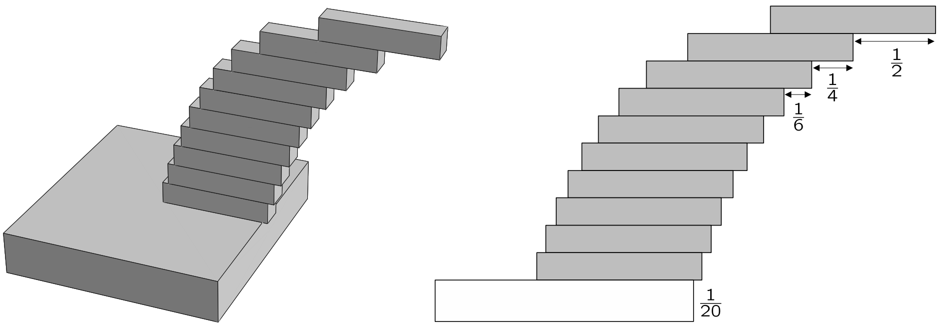

Figure 1. A harmonic stack with 10 blocks.

The classical harmonic stack of size n is composed of n blocks stacked one on top of the other, with the ith block from the top extending by $\frac{1}{2i}$ beyond the block below it. (We assume that the length of each block is 1.) The overhang achieved by the construction is clearly $\frac{1}{2}H_n$, where $H_n = \sum_{i=1}^{n}\frac{1}{i} \sim \ln n$ is the nth harmonic number. Both a 3D and a 2D view of the harmonic stack of size 10 are given in Figure 1. The harmonic stack of size n is balanced since, for every $i < n$, the center of mass of the topmost i blocks lies exactly above the right-hand edge of the $(i+1)$st block, as can be easily verified by induction. Similarly, the center of mass of all the n blocks lies exactly above the right edge of the table. A formal definition of "balanced" is given in Definition 2.1. A perhaps surprising and counterintuitive consequence of the harmonic stacks construction is that, given sufficiently many blocks, it is possible to obtain an arbitrarily large overhang!

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Harmonic stacks became widely known in the recreational math community as a result of their appearance in the *Puzzle-Math* book of Gamow and Stern [**5**, Building-Blocks, pp. 90–93] and in Martin Gardner's "Mathematical Games" section of the November 1964 issue of Scientific American [**6**] (see also [**7**, Chapter 17: Limits of Infinite Series, p. 167]). Gardner refers to the fact that an arbitrarily large overhang can be achieved, using sufficiently many blocks, as the *infinite-offset paradox*. Harmonic stacks were subsequently used by countless authors as an introduction to recurrence relations, the harmonic series, and simple optimization problems; see, e.g., [**8**, pp. 258–260]. Hall [**9**] notes that harmonic stacks started to appear in textbooks on physics and engineering mechanics as early as the mid-19th century (see, e.g., [**13**, p. 341], [**16**, pp. 140–141], [**21**, p. 183]).

It is perhaps surprising that none of the sources cited above realizes how limiting is the one-on-one restriction under which the harmonic stacks are optimal. Without this restriction, blocks can be used as counterweights, to balance other blocks. The problem then becomes vastly more interesting, and an exponentially larger overhang can be obtained.

Stacks with a specific small number of blocks that do not satisfy the one-on-one restriction were considered before by several other authors. Sutton [**20**], for example, considered the case of three blocks. One of us set a stacking problem with three uniform thin planks of lengths 2, 3, and 4 for the Archimedeans Problems Drive in 1964 [**10**]. Ainley [**1**] found the maximum overhang achievable with four blocks to be $\frac{15-4\sqrt{2}}{8} \sim 1.16789$. The optimal stacks with 3 and 4 blocks are shown, together with the corresponding harmonic stacks, in Figure 2.

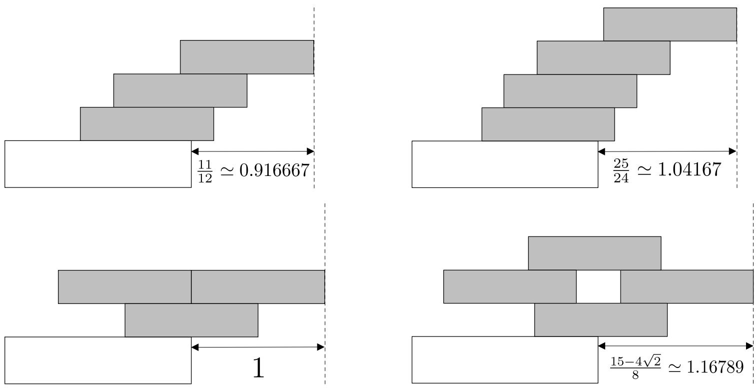

Figure 2. Optimal stacks with 3 and 4 blocks compared with the corresponding harmonic stacks.

Very recently, and independently of our work, Hall [**9**] explicitly raises the problem of finding stacks of blocks that maximize the overhang without the one-on-one restriction. (Hall calls such stacks *multiwide stacks*.) Hall gives a sequence of stacks which he claims, without proof, to be optimal. We show, however, that the stacks suggested by him are optimal only for $n \leq 19$. The stacks claimed by Hall to be optimal fall into a natural class that we call *spinal stacks*. We show in Section 3 that the maximum overhang achievable using such stacks is only $\ln n + O(1)$. Thus, although spinal stacks achieve, asymptotically, an overhang which is roughly twice the overhang achieved by harmonic stacks, they are still exponentially far from being optimal.

Optimal stacks with up to 19 blocks are shown in Figures 3 and 4. The lightly shaded blocks in these stacks form the *support set*, while the darker blocks form the *balancing set*. The *principal block* of a stack is defined to be the block which achieves

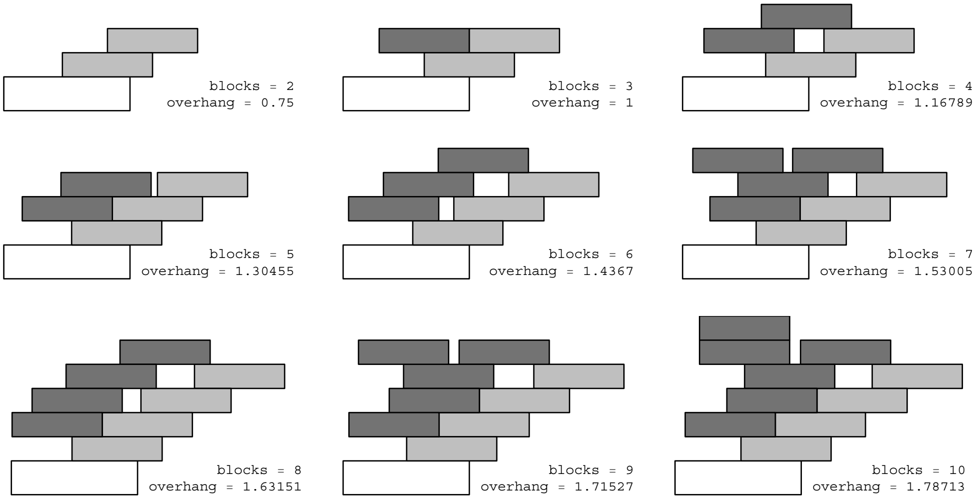

Figure 3. Optimal stacks with 2 up to 10 blocks.

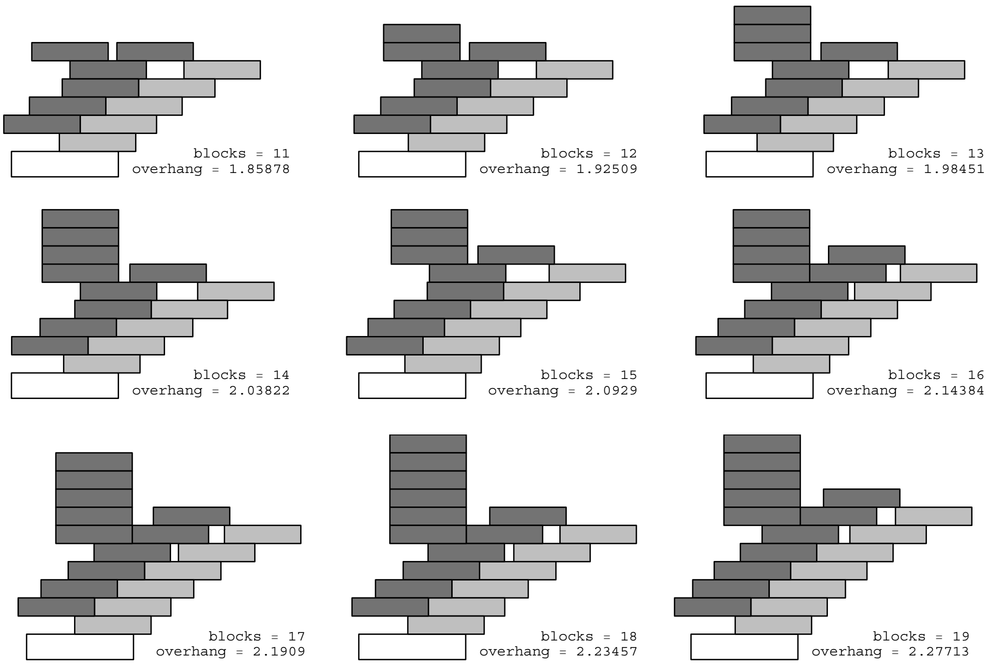

Figure 4. Optimal stacks with 11 up to 19 blocks.

the maximum overhang. (If several blocks achieve the maximum overhang, the lowest one is chosen.) The *support set* of a stack is defined recursively as follows: the principal block is in the support set, and if a block is in the support set then any block on which this block rests is also in the support set. The *balancing set* consists of all the blocks that do not belong to the support set. A stack is said to be *spinal* if its support set has a single block in each level, up to the level of the principal block. All the stacks shown in Figures 3 and 4 are thus spinal.

It is very tempting to conclude, as done by Hall [**9**], that the optimal stacks are spinal. Surprisingly, the optimal stacks for $n \geq 20$ are not spinal! Optimal stacks con-

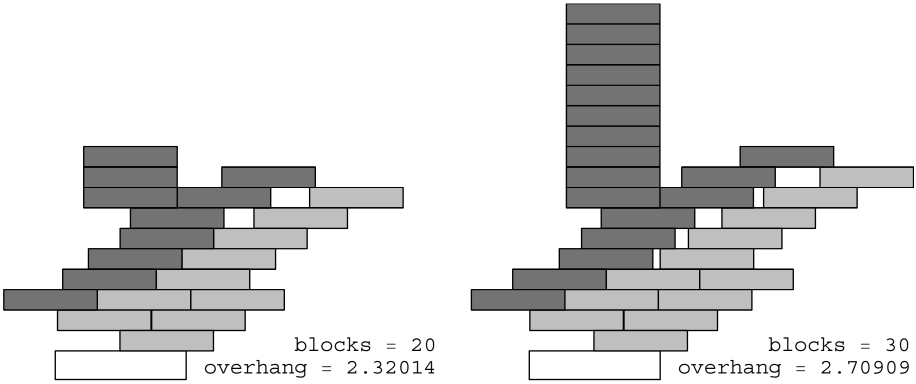

Figure 5. Optimal stacks with 20 and 30 blocks.

taining 20 and 30 blocks are shown in Figure 5. Note that the right-hand contours of these stacks are not monotone, which is somewhat counterintuitive.

For all $n \leq 30$, we have searched exhaustively through all combinatorially distinct arrangements of n blocks and found optimal displacements numerically for each of these. Some of the resulting stacks were shown in Figures 3, 4, and 5. We are confident of their optimality, though we have no formal optimality proofs, as numerical techniques were used.

While there seems to be a unique optimal placement of the blocks that belong to the support set of an optimal stack, there is usually a lot of freedom in the placement of the balancing blocks. Optimal stacks seem not to be unique for $n \geq 4$.

In view of the non-uniqueness and added complications caused by balancing blocks, it is natural to consider *loaded stacks*, which consist only of a support set with some *external forces* (or *point weights*) attached to some of their blocks. We will take the weight of each block to be 1; the size, or weight, of a loaded stack is defined to be the number of blocks contained in it plus the sum of all the point weights attached to it. The point weights are not required to be integral. Loaded stacks of weight 40, 60, 80, and 100, which are believed to be close to optimal, are shown in Figure 6. The stack of weight 100, for example, contains 49 blocks in its support set. The sum of all the external forces applied to these blocks is 51. As can be seen, the stacks become more and more non-spinal. It is also interesting to note that the stacks of Figure 6 contain small gaps that seem to occur at irregular positions. (There is also a scarcely visible gap between the two blocks at the second level of the 20-block stack of Figure 5.)

That harmonic stacks are balanced can be verified using simple center-of-mass considerations. These considerations, however, are not enough to verify the balance of more complicated stacks, such as those in Figures 3, 4, 5, and 6. A formal mathematical definition of "balanced" is given in the next section. Briefly, a stack is said to be balanced if there is an appropriate set of forces acting between the blocks of the stacks, and between the blocks at the lowest level and the table, under which all blocks are *in equilibrium*. A block is in equilibrium if the sum of the forces and the sum of the moments acting upon it are both 0. As shown in the next section, the balance of a given stack can be determined by checking whether a given set of *linear inequalities* has a feasible solution.

Given the fact that the 3-block stack that achieves the maximum overhang is an *inverted 2-triangle* (see Figure 2), it is natural to enquire whether larger inverted triangles are also balanced. Unfortunately, the next inverted triangle is already unbalanced and

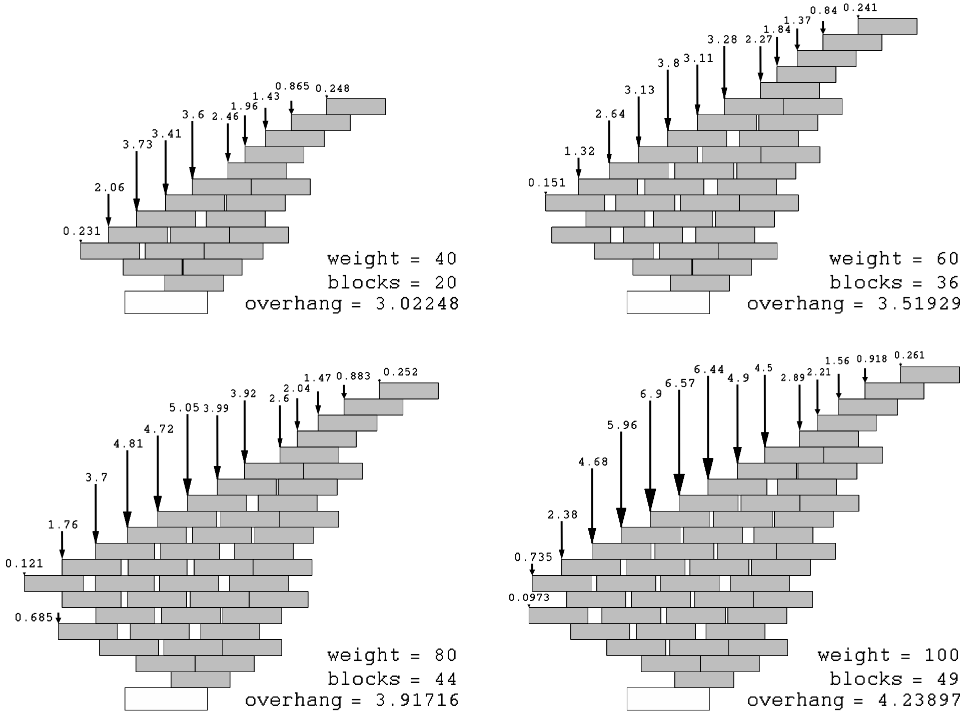

Figure 6. Loaded stacks, believed to be close to optimal, of weight 40, 60, 80, and 100.

would collapse in the way indicated in Figure 7. Inverted triangles show that simple center-of-mass considerations are not enough to determine the balance of stacks. As an indication that balance issues are not always intuitive, we note that inverted triangles are falsely claimed by Jargodzki and Potter [**11**, Challenge 271: A staircase to infinity, p. 246] to be balanced.

Another appealing structure, the m-*diamond*, illustrated for $m = 4$ and 5 in Figure 8, consists of a symmetric diamond shape with rows of length $1, 2, \ldots, m-1, m,$

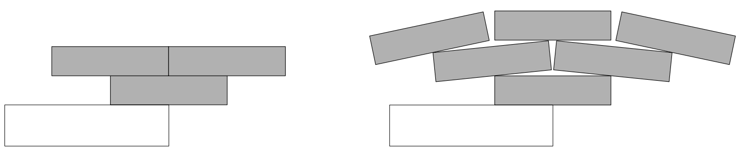

Figure 7. The balanced inverted 2-triangle and the unbalanced inverted 3-triangle.

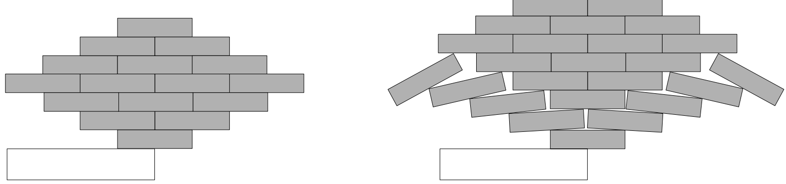

Figure 8. The balanced 4-diamond and the unbalanced 5-diamond.

$m-1, \ldots, 2, 1$. Small diamonds were considered by Drummond [**3**]. The m-diamond uses m^2 blocks and would give an overhang of $m/2$, but unfortunately it is unbalanced for $m \geq 5$. A 5-diamond would collapse in the way indicated in the figure. An m-diamond could be made balanced by adding a column of sufficiently many blocks resting on the top block. The methodology introduced in Section 3 can be used to show that, for $m \geq 5$, a column of at least $2^m - m^2 - 1$ blocks would be needed. We can show that this number of blocks is also sufficient, giving a stack of $2^m - 1$ blocks with an overhang of $m/2$. It is interesting to note that these stacks are already better than the classical harmonic stacks, as with $n = 2^m - 1$ blocks they give an overhang of $\frac{1}{2}\log_2(n+1) \simeq 0.693 \ln n$.

Determining the *exact* overhang achievable using n blocks, for large values of n, seems to be a formidable task. Our main goal in this paper is to determine the *asymptotic growth* of this quantity. Our main result is that there exists a constant $c > 0$ such that an overhang of $cn^{1/3}$ is achievable using n blocks. Note that this is an exponential improvement over the $\frac{1}{2}\ln n + O(1)$ overhang of harmonic stacks and the $\ln n + O(1)$ overhang of the best spinal stacks! In a subsequent paper [**15**], with three additional coauthors, we show that our improved stacks are asymptotically optimal, i.e., there exists a constant $C > 0$ such that the overhang achievable using n blocks is at most $Cn^{1/3}$.

Our stacks that achieve an asymptotic overhang of $cn^{1/3}$, for some $c > 0$, are quite simple. We construct an explicit sequence of stacks, called *parabolic stacks*, with the rth stack in the sequence containing about $2r^3/3$ blocks and achieving an overhang of $r/2$. One stack in this sequence is shown in Figure 9. The balance of the parabolic stacks is established using an inductive argument.

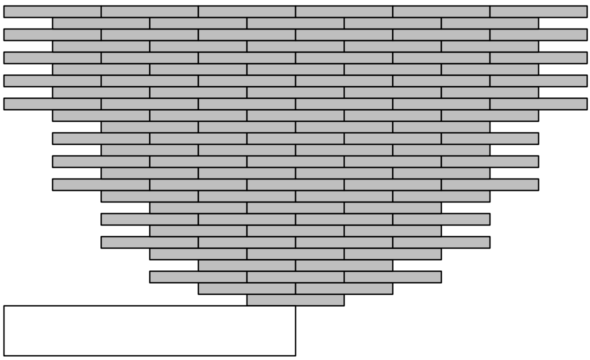

Figure 9. A parabolic stack consisting of 111 blocks and giving an overhang of 3.

The remainder of this paper is organized as follows. In the next section we give formal definitions of all the notions used in this paper. In Section 3 we analyze *spinal stacks*. In Section 4, which contains our main results, we introduce and analyze our parabolic stacks. In Section 5 we describe some experimental results with stacks that seem to improve, by a constant factor, the overhang achieved by parabolic stacks. We end in Section 6 with some open problems.

2. STACKS AND THEIR BALANCE. As the maximum overhang problem is physical in nature, our first task is to formulate it mathematically. We consider a 2-dimensional version of the problem. This version captures essentially all the interesting features of the overhang problem.

A *block* is a rectangle of length 1 and height h with uniform density and unit weight. (We shall see shortly that the height h is unimportant.) We assume that the table occupies the quadrant $x, y \leq 0$ of the 2-dimensional plane. A *stack* is a collection of blocks. We consider only *orthogonal* stacks in which the sides of the blocks are parallel to the axes, with the length of each block parallel to the x-axis. The position of a block is then determined by the coordinate (x, y) of its lower left corner. Such a block occupies the box $[x, x+1] \times [y, y+h]$. A stack composed of n blocks is specified by the sequence $(x_1, y_1), \ldots, (x_n, y_n)$ of the coordinates of the lower left corners of its blocks. We require each y_i to be a nonnegative integral multiple of h, the height of the blocks. Blocks can touch each other but are not allowed to overlap. The *overhang* of the stack is $1 + \max_{i=1}^{n} x_i$.

A block at position (x_1, y_1) *rests on* a block in position (x_2, y_2) if $|x_1 - x_2| < 1$ and $y_1 - y_2 = h$. The *interval of contact* between the two blocks is then $[\max\{x_1, x_2\}, 1 + \min\{x_1, x_2\}] \times \{y_1\}$. A block placed at position $(x, 0)$ *rests on the table* if $x < 0$. The interval of contact between the block and the table is $[x, \min\{x+1, 0\}] \times \{0\}$.

When block A rests on block B, the two blocks may exert a (possibly infinitesimal) force on each other at every point along their interval of contact. A force is a vector acting at a specified point. By Newton's third law, forces come in opposing pairs. If a force f is exerted on block A by block B, at (x, y), then a force $-f$ is exerted on block B by block A, again at (x, y). We assume that edges of all the blocks are completely smooth, so that there is no *friction* between them. All the forces exerted on block A by block B, and vice versa, are therefore *vertical* forces. Furthermore, as there is nothing that holds the blocks together, blocks A and B can *push*, but not pull, one another. Thus, if block A rests on block B, then all the forces applied on block A by block B point upward, while all the forces applied on block B by block A point downward, as shown on the left in Figure 10. Similar forces are exerted between the table and the blocks that rest on it.

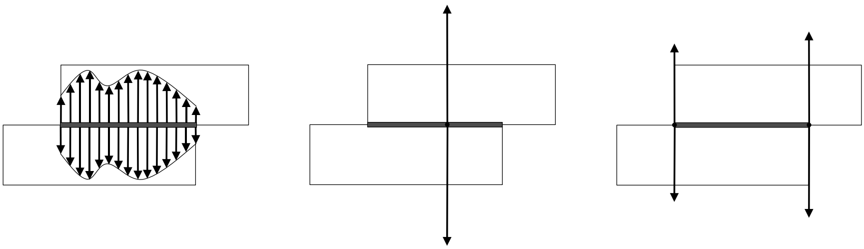

Figure 10. Equivalent sets of forces acting between two blocks.

The distribution of forces acting between two blocks may be hard to describe explicitly. Since all these forces point in the same direction, they can always be replaced by a single *resultant* force acting at some point within their interval of contact, as shown in the middle drawing of Figure 10. As an alternative, they may be replaced by two resultant forces that act at the endpoints of the contact interval, as shown on the right in Figure 10. Forces acting between blocks and between the blocks and the table are said to be *internal* forces.

Each block is also subjected to a downward *gravitational force* of unit size, acting at its center of mass. As the blocks are assumed to be of uniform density, the center of mass of a block whose lower left corner is at (x, y) is at $(x + \frac{1}{2}, y + \frac{h}{2})$.

A rigid body is said to be in *equilibrium* if the sum of the forces acting on it, and the sum of the *moments* they apply on it, are both zero. A 2-dimensional rigid body acted upon by k vertical forces $f_1, f_2, \ldots, f_k$ at $(x_1, y_1), \ldots, (x_k, y_k)$ is in equilibrium if and only if $\sum_{i=1}^k f_i = 0$ and $\sum_{i=1}^k x_i f_i = 0$. (Note that $f_1, f_2, \ldots, f_k$ are *scalars* that represent the magnitudes of vertical forces.)

A collection of internal forces acting between the blocks of a stack, and between the blocks and the table, is said to be a *balancing* set of forces if the forces in this collection satisfy the requirements mentioned above (i.e., all the forces are vertical, they come in opposite pairs, and they act only between blocks that rest on each other) and if, taking into account the gravitational forces acting on the blocks, all the blocks are in equilibrium under this collection of forces. We are now ready for a formal definition of balance.

Definition 2.1 (Balance). A stack of blocks is *balanced* if and only if it admits a balancing set of forces.

Static balance problems of the kind considered here are often *under-determined*, so that the resultants of balancing forces acting between the blocks are usually not uniquely determined. It was the consideration by one of us of balance issues that arise in the game of *Jenga* [**22**] which stimulated this current work. The following theorem shows that the balance of a given stack can be checked efficiently.

Theorem 2.2. *The balance of a stack containing n blocks can be decided by checking the feasibility of a collection of linear equations and inequalities with $O(n)$ variables and constraints.*

Proof. Let $(x_1, y_1), \ldots, (x_n, y_n)$ be the coordinates of the lower left corners of the blocks in the stack. Let B_i, for $1 \le i \le n$, denote the ith block of the stack, and let B_0 denote the table. Let B_i/B_j, where $0 \le i, j \le n$, signify that B_i rests on B_j. If B_i/B_j, we let $a_{ij} = \max\{x_i, x_j\}$ and $b_{ij} = \min\{x_i, x_j\} + 1$ be the x-coordinates of the endpoints of the interval of contact between blocks i and j. (If $j = 0$, then $a_{i0} = x_i$ and $b_{i0} = \min\{x_i + 1, 0\}$.)

For all i and j such that B_i/B_j, we introduce two variables f_{ij}^0 and f_{ij}^1 that represent the resultant forces that act between B_i and B_j at a_{ij} and b_{ij}. By Definition 2.1 and the discussion preceding it, the stack is balanced if and only if there is a feasible solution to the following set of linear equalities and inequalities:

$$\sum_{j\,:\,B_i/B_j} (f_{ij}^0 + f_{ij}^1) - \sum_{k\,:\,B_k/B_i} (f_{ki}^0 + f_{ki}^1) = 1, \qquad \text{for } 1 \le i \le n;$$

$$\sum_{j\,:\,B_i/B_j} (a_{ij} f_{ij}^0 + b_{ij} f_{ij}^1) - \sum_{k\,:\,B_k/B_i} (a_{ki} f_{ki}^0 + b_{ki} f_{ki}^1) = x_i + \frac{1}{2}, \qquad \text{for } 1 \le i \le n;$$

$$f_{ij}^0, f_{ij}^1 \ge 0, \qquad \text{for } i, j \text{ such that } B_i/B_j.$$

The first $2n$ equations require the forces applied on the blocks to exactly cancel the forces and moments exerted on the blocks by the gravitational forces. (Note that the table is not required to be in equilibrium.) The inequalities $f_{ij}^0, f_{ij}^1 \ge 0$, for every i and j such that B_i/B_j, require the forces applied on B_i by B_j to point upward. As a unit length block can rest on at most two other unit length blocks, the number of variables is at most $4n$ and the number of constraints is therefore at most $6n$. The feasibility of such a system of linear equations and inequalities can be checked using *linear programming* techniques. (See, e.g., Schrijver [**17**].) ■

Definition 2.3 (Maximum overhang, the function $D(n)$). The maximum overhang that can be achieved using a balanced stack comprising n blocks of length 1 is denoted by $D(n)$.

We now repeat the definitions of the principal block, the support set, and the balancing set sketched in the introduction.

Definitions 2.4 (Principal block, support set, balancing set). The block of a stack that achieves the maximum overhang is the *principal block* of the stack. If several blocks achieve the maximum overhang, the lowest one is chosen. The *support set* of a stack is defined recursively as follows: the principal block is in the support set, and if a block is in the support set then any block on which this block rests is also in the support set. The *balancing set* consists of any blocks that do not belong to the support set.

The blocks of the support sets of the stacks in Figures 3, 4, and 5 are shown in light gray while the blocks in the balancing sets are shown in dark gray. The purpose of blocks in the support set, as its name indicates, is to support the principal block. The blocks in the balancing set, on the other hand, are used to counterbalance the blocks in the support set.

As already mentioned, there is usually a lot of freedom in the placement of the blocks of the balancing set. To concentrate on the more important issue of where to place the blocks of the support set, it is useful to introduce the notion of loaded stacks.

Definitions 2.5 (Loaded stacks, the function $D^*(w)$). A *loaded stack* consists of a set of blocks with some *point weights* attached to them. The *weight* of a loaded stack is the sum of the weights of all the blocks and point weights that participate in it, where the weight of each block is taken to be 1. A loaded stack is said to be balanced if it admits a balancing set of forces, as for unloaded stacks, but now also taking into account the point weights. The maximum overhang that can be achieved using a balanced loaded stack of weight w is denoted by $D^*(w)$.

Clearly $D^*(n) \geq D(n)$, as a standard stack is a trivially loaded stack with no point weights. When drawing loaded stacks, as in Figure 6, we depict point weights as external forces acting on the blocks of the stack, with the length of the arrow representing the force proportional to the weight of the point weight. (Since forces can be transmitted vertically downwards through any block, we may assume that point weights are applied only to upper edges of blocks outside any interval of contact.)

As the next lemma shows, balancing blocks can always be replaced by point weights, yielding loaded stacks in which all blocks belong to the support set.

Lemma 2.6. *For every balanced stack that contains k blocks in its support set and $n - k$ blocks in its balancing set, there is a balanced loaded stack composed of k blocks, all in the support set, and additional point weights of total weight $n - k$ that achieves the same overhang.*

Proof. Consider the set of forces exerted on the support set of the stack by the set of balancing blocks. From the definition of the support set, no block of the support set can rest on any balancing block; therefore the effect of the balancing set can be represented by a set of *downward* vertical forces on the support set, or equivalently by a finite set of point weights attached to the support set, with the same total weight as the set of balancing blocks. ■

Given a loaded stack of integral weight, it is in many cases possible to replace the set of point weights by a set of appropriately placed balancing blocks. In some cases, however, such a conversion of a loaded stack into a standard stack is not possible. The optimal loaded stacks of weight 3, 5, and 7 cannot be converted into standard stacks without decreasing the overhang, as the number of point weights needed is larger than the number of blocks remaining. (The cases of weights 3 and 5 are shown in Figure 11.) In particular, we get that $D^*(3) = \frac{11-2\sqrt{6}}{6} > D(3) = 1$. Experiments with optimal loaded stacks lead us, however, to conjecture that the difference $D^*(n) - D(n)$ tends to 0 as n tends to infinity.

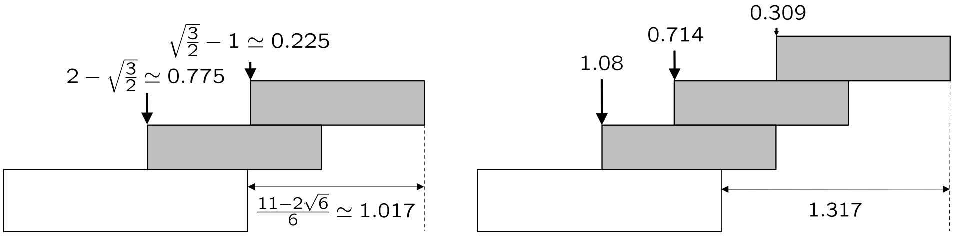

Figure 11. Optimal loaded stacks of weight 3 and 5.

Conjecture 2.7. $D(n) = D^*(n) - o(1)$.

3. SPINAL STACKS. In this section we focus on a restricted, but quite natural, class of stacks which admits a fairly simple analysis.

Definitions 3.1 (Spinal stacks, spine). A stack is *spinal* if its support set has just a single block at each level. The support set of a spinal stack is referred to as its *spine*.

The optimal stacks with up to 19 blocks, depicted in Figures 3 and 4, are spinal. The stacks of Figure 5 are *not* spinal. A stack is said to be *monotone* if the x-coordinates of the rightmost blocks in the various levels, starting from the bottom, form an increasing sequence. It is easy to see that every monotone stack is spinal.

Definitions 3.2 (The functions $S(n)$, $S^*(w)$, and $S_k^*(w)$). Let $S(n)$ be the maximum overhang achievable using a spinal stack of size n. Similarly, let $S^*(w)$ be the maximum overhang achievable using a loaded spinal stack of weight w, and let $S_k^*(w)$ be the maximum overhang achievable using a spinal stack of weight w with exactly k blocks in its spine.

It is tempting to make the (false) assumption that optimal stacks are spinal. (As mentioned in the introduction, this assumption is implicit in [**9**].) The assumption holds, however, only for $n \leq 19$. (See the discussion following Theorem 4.4.) As spinal stacks form a very natural class of stacks, it is still interesting to investigate the maximum overhang achievable using stacks of this class.

A generic loaded spinal stack with k blocks in its spine is shown in Figure 12. We denote the blocks from top to bottom as $B_1, B_2, \ldots, B_k$, with B_1 being the principal

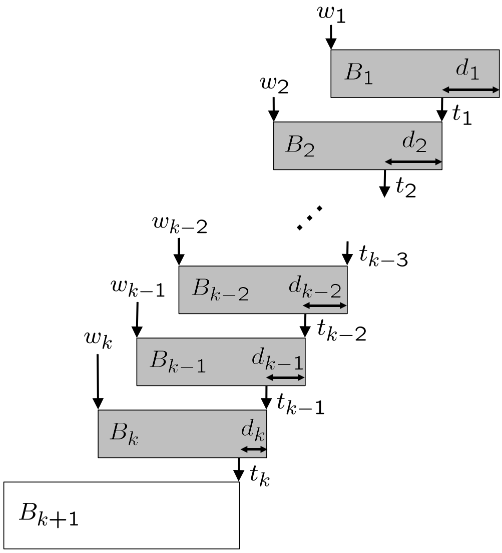

Figure 12. A generic loaded spinal stack.

block. We regard the tabletop as B_{k+1}. For $1 \le i \le k$, the weight attached to the left edge of B_i is denoted by w_i, and the relative overhang of B_i beyond B_{i+1} is denoted by d_i. We define $t_i = \sum_{j=1}^{i}(1 + w_j)$, the total downward force exerted upon B_{i+1} by block B_i. We also define $t_0 = 0$. Note that $t_i = t_{i-1} + w_i + 1$, for $1 \le i \le k$, and that $t_k = w = k + \sum_{i=1}^{k} w_i$, the total weight of the loaded stack.

The assumptions made in Figure 12, that each block is supported by a force that acts along the right-hand edge of the block underneath it and that all point weights are attached to the left-hand ends of blocks, are justified by the following lemma.

Lemma 3.3. *In an optimal loaded spinal stack:* (i) *Each block is supported by a force acting along the right-hand edge of the block underneath it. In particular, the stack is monotone.* (ii) *All point weights are attached to the left-hand ends of blocks.*

Proof. For (i), suppose there were some block B_{i+1} $(1 \le i \le k)$ where the resultant force exerted on it from B_i does not go through its right-hand end. If $i < k$ then B_{i+1} could be shifted some distance to the left and B_i together with all the blocks above it shifted to the right in such a way that the resultant force from B_{i+1} on B_{i+2} remains unchanged in position and the stack is still balanced. In the case of $i = k$ (where B_{k+1} is the tabletop), the whole stack could be moved to the right. The result of any such change is a balanced spinal stack with an increased overhang, a contradiction. As an immediate consequence, we get that optimal spinal stacks are monotone.

For (ii), suppose that some block has weights attached other than at its left-hand end. We may replace all such weights by the same total weight concentrated at the left end. The result will be to move the resultant force transmitted to any lower block somewhat to the left. Since the stack is monotone, this change cannot unbalance the stack, and indeed would then allow the overhang to be increased by slightly shifting all blocks to the right; again a contradiction. ∎

We next note that for any nonnegative point weights $w_1, w_2, \dots, w_k \ge 0$, there are appropriate positive displacements $d_1, d_2, \dots, d_k > 0$ for which the generic spinal stack of Figure 12 is balanced.

Lemma 3.4. *A loaded spinal stack with k blocks in its spine that satisfies the two conditions of Lemma 3.3 is balanced if and only if*

$$d_i = \frac{w_i + \frac{1}{2}}{t_i} = 1 - \frac{t_{i-1} + \frac{1}{2}}{t_i},$$

for $1 \le i \le k$.

Proof. The lemma is verified using a simple calculation. The net downward force acting on B_i is $(w_i + t_{i-1} + 1) - t_i = 0$, by the definition of t_i. (Recall that $t_i = \sum_{j=1}^{i}(1 + w_j)$.) The net moment acting on B_i, computed relative to the right-hand edge of B_i, is $d_i t_i - (\frac{1}{2} + w_i)$, which vanishes if and only if

$$d_i = \frac{\frac{1}{2} + w_i}{t_i} = 1 - \frac{t_{i-1} + \frac{1}{2}}{t_i},$$

as required. ■

Note, in particular, that if $w_i = 0$ for $1 \le i \le k$, then $t_i = i$ and $d_i = \frac{1}{2i}$, and we are back to the classic harmonic stacks.

We can now also justify the claim made in the introduction concerning the unbalanced nature of diamond stacks. Consider the spine of an m-diamond. In this case, $d_i = \frac{1}{2}$ for all i and so the balance conditions give the equations $t_i = 2t_{i-1} + 1$ for $1 \le i \le m$. As $t_0 = 0$, we have $t_i \ge 2^i - 1$ for all i and hence $t_m \ge 2^m - 1$. Since t_m is the total weight of the stack, the number of extra blocks required to be added for balance is at least $2^m - 1 - m^2$, which is positive for $m \ge 5$.

Next, we characterize the choice of the weights $w_1, w_2, \ldots, w_k$, or alternatively of the total loads $t_1, t_2, \ldots, t_k$, that maximizes the overhang achieved by a spinal stack of total weight w. (Note that $w_i = t_i - t_{i-1} - 1$, for $1 \le i \le k$.)

Lemma 3.5. *If a loaded spinal stack with total weight w and with k blocks in its spine achieves the maximal overhang of $S_k^*(w)$, then for some j $(1 \le j \le k)$ we have $t_i^2 = (t_{i-1} + \frac{1}{2})t_{i+1}$, for $1 \le i < j$, and $w_i = 0$, for $j < i \le k$.*

Proof. Let $w_1, w_2, \ldots, w_k$ be the point weights attached to the blocks of an optimal spinal stack with overhang $S_k^*(w)$. For some i satisfying $1 \le i < k$ and a small x, consider the stack obtained by increasing the point weight at the left-hand end of block B_i from w_i to $w_i + x$, and decreasing the point weight on B_{i+1} from w_{i+1} to $w_{i+1} - x$, assuming that $w_{i+1} \ge x$. Note that this small perturbation does not change the total weight of the stack. The overhang of the perturbed stack is

$$V(x) = \left(1 - \frac{t_{i-1} + \frac{1}{2}}{t_i + x}\right) + \frac{w_{i+1} - x + \frac{1}{2}}{t_{i+1}} + \sum_{j \ne i, i+1} \frac{w_j + \frac{1}{2}}{t_j}.$$

The first two terms in the expression above are the new displacements $d_i(x)$ and $d_{i+1}(x)$. Note that all other displacements are unchanged. Differentiating $V(x)$ we get

$$V'(x) = \frac{t_{i-1} + \frac{1}{2}}{(t_i + x)^2} - \frac{1}{t_{i+1}} \quad \text{and} \quad V'(0) = \frac{t_{i-1} + \frac{1}{2}}{t_i^2} - \frac{1}{t_{i+1}}.$$

If $w_i = 0$ while $w_{i+1} > 0$, then $t_{i-1} = t_i - 1$ and $t_{i+1} > t_i + 1$, which in conjunction with $t_i \geq 1$ implies that $V'(0) > 0$, contradicting the optimality of the stack. Thus, if in an optimal stack we have $w_i = 0$, then also $w_{i+1} = w_{i+2} = \cdots = w_k = 0$. If $w_i, w_{i+1} > 0$, then we must have $V'(0) = 0$, or equivalently $t_i^2 = (t_{i-1} + \frac{1}{2})t_{i+1}$, as claimed. ∎

The optimality equations given in Lemma 3.5 can be solved numerically to obtain the values of $S_k^*(w)$ for specific values of w and k. The value of $S^*(w)$ is then found by optimizing over k. The optimal loaded spinal stacks of weight 3 and 5, which also turn out to be the optimal loaded stacks of these weights, are shown in Figure 11. The optimality equations of Lemma 3.5 were also used to compute the spines of the optimal stacks with up to 19 blocks shown in Figures 3 and 4. The spines of the stacks with 3 and 5 blocks were obtained by adding the requirement that no point weight be attached to the topmost block of the spine. A somewhat larger example is given on the top left of Figure 14 where the optimal loaded spinal stack of weight 100 is shown. It is interesting to note that the point weights in optimal spinal stacks form an almost arithmetical progression. This observation is used in the proof of Theorem 3.8.

Numerical experiments suggest that for every $w \geq 1$, all the point weights in the spinal stacks with overhang $S^*(w)$ are nonzero. There are, however, non-optimal values of k for which some of the bottom blocks in the stack that achieves an overhang of $S_k^*(w)$ have no point weights attached to them. We next show, without explicitly using the optimality conditions of Lemma 3.5, that $S^*(w) = \ln w + \Theta(1)$.

Theorem 3.6. $S^*(w) < \ln w + 1$.

Proof. For fixed total weight $w = t_k$ and fixed k, the largest possible overhang $S_k^*(w) = \sum_{i=1}^k d_i$ is attained when the conditions of Lemmas 3.3 and 3.4 (and 3.5) hold. Thus, as $t_0 = 0$,

$$\sum_{i=1}^k d_i = \sum_{i=1}^k \left(1 - \frac{t_{i-1} + \frac{1}{2}}{t_i}\right) < k - \sum_{i=2}^k \frac{t_{i-1}}{t_i}.$$

Putting $x_i = \frac{t_{i-1}}{t_i}$, we see that

$$S_k^*(w) < k - \sum_{i=2}^k x_i \quad \text{and} \quad \prod_{i=2}^k x_i = \frac{t_1}{t_k} \geq \frac{1}{w}.$$

The minimum sum for a finite set of positive real numbers with fixed product is attained when the numbers are equal, hence

$$S_k^*(w) < k - (k-1)w^{-\frac{1}{k-1}}.$$

Let $z = \frac{k-1}{\ln w}$, so that $k - 1 = z \ln w$ and $w^{-\frac{1}{k-1}} = e^{-1/z}$. Then

$$S_k^*(w) < 1 + z \ln w (1 - e^{-1/z}) < 1 + \ln w,$$

as $z(1 - e^{-1/z}) \leq 1$, for every $z > 0$. ∎

Corollary 3.7. $S(n) < \ln n + 1$.

We can now describe a construction of loaded spinal stacks which achieves an overhang agreeing asymptotically with the upper bound proved in Theorem 3.6.

Theorem 3.8. $S^*(w) > \ln w - 1.313$.

Proof. We construct a spine with $k = \lfloor \sqrt{w} \rfloor$ blocks in it with $w_i = 2(i-1)$, for $1 \le i \le k$. It follows easily by induction that $t_i = i^2$, for $1 \le i \le k$. In particular, the total weight of the stack is $t_k = k^2 \le w$, as required. By Lemma 3.4, we get that

$$d_i = \frac{w_i + \frac{1}{2}}{t_i} = \frac{2(i-1) + \frac{1}{2}}{i^2} = \frac{2}{i} - \frac{3}{2i^2}.$$

Thus,

$$S^*(w) \ge \sum_{i=1}^{k} d_i = 2\sum_{i=1}^{k} \frac{1}{i} - \frac{3}{2}\sum_{i=1}^{k}\frac{1}{i^2} = 2H_{\lfloor\sqrt{w}\rfloor} - \frac{3}{2}\sum_{i=1}^{\lfloor\sqrt{w}\rfloor}\frac{1}{i^2}$$
$$> \ln w + 2\gamma - \frac{\pi^2}{4} > \ln w - 1.313.$$

In the above inequality, $\gamma \simeq 0.5772156$ is Euler's gamma. ∎

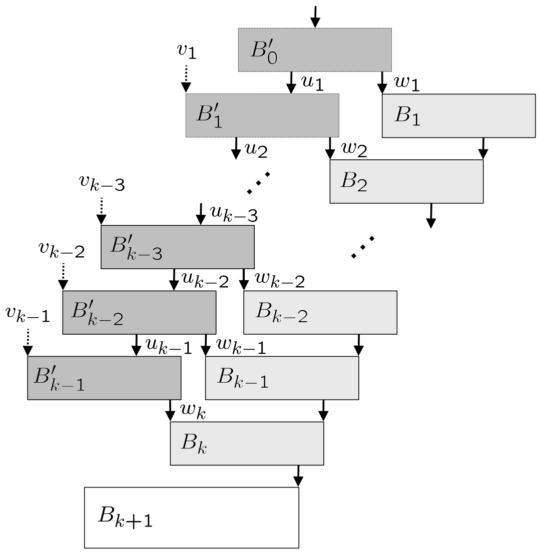

Figure 13. A spinal stack with a shield.

We next discuss a technique that can be used to convert loaded spinal stacks into standard stacks. This is of course done by constructing balancing sets that apply the required forces on the left-hand edges of the spine blocks. The first step is the placement of *shield* blocks on top of the spine blocks, as shown in Figure 13. We let B'_i, for $0 \le i \le k-1$, be the shield block placed on top of spine block B_{i+1} and alongside spine block B_i for $i > 0$. We let y_i be the x-coordinate of the left edge of B'_i, for $1 \le i \le k-1$. Note that $x_{i+1} - 1 < y_i \le x_i - 1$, where x_i is the x-coordinate of the left edge of B_i.

Shield block B_i' applies a downward force of w_{i+1} on B_{i+1}. The force is applied at x_{i+1}, i.e., at the left edge of B_{i+1}. Block B_i' also applies a downward force of u_{i+1} on B_{i+1}' at z_{i+1}, where $y_i \leq z_{i+1} \leq y_{i+1} + 1$. Similarly, block B_{i-1}' applies a downward force of u_i on B_i' at z_i. Finally a downward external force of v_i is applied on the left edge of B_i'. The goal of the shield blocks is to aggregate the forces that should be applied on the spine blocks and to replace them by a set of fewer *integral* forces that are to be applied on the shield blocks. We will therefore place the shield blocks and choose the forces u_i and their positions in such a way that most of the v_i will be 0. (This is why we use dashed arrows to represent the v_i forces in Figure 13.)

The shield blocks are in equilibrium provided that the following balance conditions are satisfied:

$$u_i + v_i + 1 = u_{i+1} + w_{i+1},$$

$$z_i u_i + y_i v_i + (y_i + \tfrac{1}{2}) = z_{i+1} u_{i+1} + x_{i+1} w_{i+1},$$

for $1 \leq i \leq k-1$. (We define $u_k = 0$.) It is easy to see that if $u_{i+1}, w_{i+1}, x_{i+1}, y_{i+1}$, and z_{i+1} are set, then any choice of v_i uniquely determines u_i and z_i. The choice is feasible if $u_i, v_i \geq 0$ and $y_{i-1} \leq z_i$.

In our constructions, we used the following heuristic to place the shield blocks and specify the forces between them. We start placing the shield blocks from the bottom up. In most cases, we choose $y_i = x_i - 1$ and $v_i = 0$, i.e., B_i' is adjacent to B_i and no external force is applied to it. Eventually, however, it may happen that $z_{i+1} < x_i - 1$, which makes it impossible to place B_i' adjacent to B_i and still apply the force u_{i+1} down on B_{i+1} at z_{i+1}. In that case we choose $y_i = z_{i+1}$. A more significant event, that usually occurs soon after the previous event, is when $z_{i+1} \leq x_{i+1} - 1$, in which case no placement of B_i' allows it to apply the forces u_{i+1} and v_{i+1} on B_{i+1}' and B_{i+1} at the required positions, as they are at least a unit distance apart. In this case, we introduce a nonzero, integral, external force v_{i+1} as follows. We let $v_{i+1} = \lfloor (1 - z_{i+1} + y_{i+1}) u_{i+1} \rfloor$ and then recompute u_{i+1} and z_{i+1}. It is easy to check that $u_{i+1}, v_{i+1} \geq 0$ and that $y_i \leq z_{i+1} \leq y_{i+1} + 1$. If we now have $z_{i+1} > x_{i+1} - 1$, then the process can continue. Otherwise we stop. In our experience, we were always able to use this process to place all the shield blocks, except for a very few top ones. The v_i forces left behind tend to be few and far apart. When this process is applied, for example, on the optimal loaded spinal stack of weight 100, only one such external force is needed, as shown in the second diagram of Figure 14.

The nonzero v_i's can be easily realized by erecting appropriate towers, as shown at the bottom of Figure 14. The top part of the balancing set is then designed by solving a small linear program. We omit the fairly straightforward details.

The overhang achieved by the spinal stack shown at the bottom of Figure 14 is about 3.6979, which is a considerable improvement on the 2.5937 overhang of a 100-block harmonic stack, but is also substantially less than the 4.23897 overhang of the non-spinal loaded stack of weight 100 given in Figure 6.

Using the heuristic described above we were able to fit appropriate balancing sets for all optimal loaded spinal stacks of integer weight n, for every $n \leq 1000$, with the exception of $n = 3, 5, 7$. We conjecture that the process succeeds for every $n \neq 3, 5, 7$.

Conjecture 3.9. *$S(n) = S^*(n)$ for $n \neq 3, 5$, or 7.*

4. PARABOLIC STACKS.

We now give a simple explicit construction of n-block stacks with an overhang of about $(3n/16)^{1/3}$, an *exponential* improvement over the

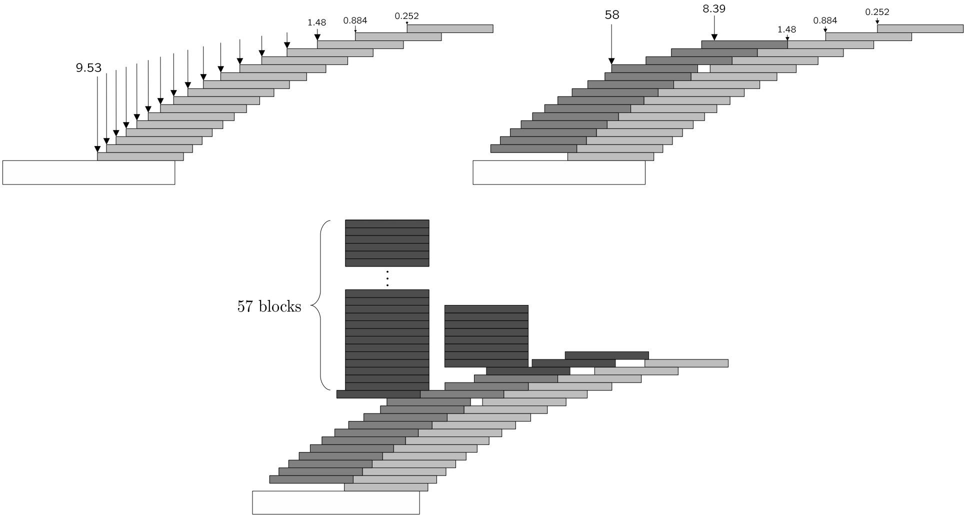

Figure 14. Optimal loaded spinal stack of weight 100 (top left), with shield added (top right) and with a complete balancing set added (bottom).

$O(\log n)$ overhang achievable using spinal stacks in general and the harmonic stacks in particular. Though the stacks of this sequence are not optimal (see the empirical results of the next section), they are within a constant factor of optimality, as will be shown in a subsequent paper [**15**].

The stacks constructed in this section are what we term *brick-wall* stacks. The blocks in each row are contiguous, and each is centered over the ends of blocks in the row beneath. This resembles the simple "stretcher-bond" pattern in real-life bricklaying. Overall the stacks have a symmetric roughly parabolic shape, hence the name, with vertical axis at the table edge and a brick-wall structure. An illustration of a 111-block *parabolic* 6-stack with overhang 3 was given in Figure 9.

An r-*row* is a row of r adjacent blocks, symmetrically placed with respect to $x = 0$. An r-*slab*, for $r \geq 2$, has height $2r - 3$ and consists of alternating r-rows and $(r-1)$-rows, starting and finishing with r-rows. An r-slab therefore contains $r(r-1) + (r-1)(r-2) = 2(r-1)^2$ blocks. Figure 15 shows r-slabs, for $r = 2, 3, \ldots, 6$. A

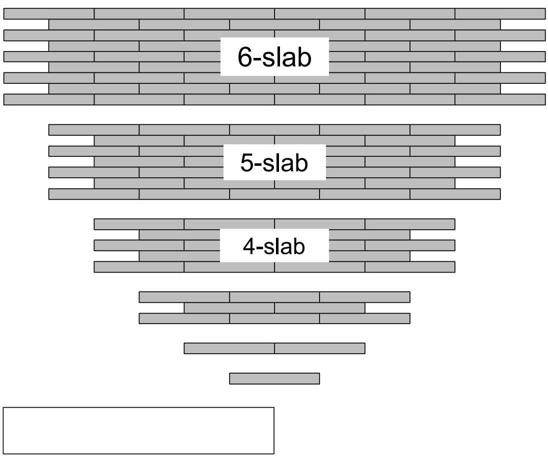

Figure 15. A 6-stack composed of r-slabs, for $r = 2, 3, \ldots, 6$, and an additional block.

parabolic d-stack, or just *d-stack*, for short, is a d-slab on a $(d-1)$-slab on $\ldots$ on a 2-slab on a single block. The slabs shown in Figure 15 thus compose a 6-stack.

Lemma 4.1. *A parabolic d-stack contains $\frac{d(d-1)(2d-1)}{3}+1$ blocks and, if balanced, has an overhang of $\frac{d}{2}$.*

Proof. The number of blocks contained in a d-stack is

$$1+\sum_{r=2}^{d} 2(r-1)^2 = 1+\frac{d(d-1)(2d-1)}{3}.$$

The overhang achieved, if the stack is balanced, is half the width of the top row, i.e., $\frac{d}{2}$. ■

In preparation for proving the balance of parabolic stacks, we show in the next lemma that a slab can concentrate a set of forces acting on its top together with the weights of its own blocks down into a narrower set of forces acting on the row below it. The lemma is illustrated in Figure 16.

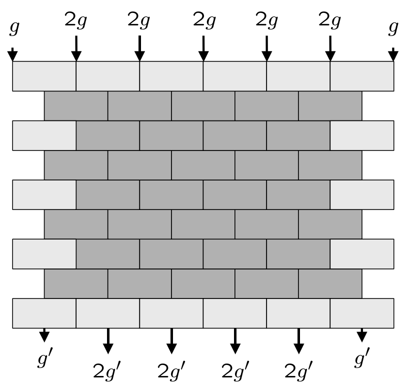

Figure 16. A 6-slab with a grey 5-slab contained in it.

Lemma 4.2. *For any $g \geq 0$, an r-slab with forces of $g, 2g, 2g, \ldots, 2g, g$ acting downwards onto its top row at positions*

$$-\frac{r}{2}, -\frac{r-2}{2}, -\frac{r-4}{2}, \ldots, \frac{r-2}{2}, \frac{r}{2},$$

respectively, can be balanced by applying a set of upward forces $g', 2g', 2g', \ldots, 2g', g'$, where $g' = \frac{r}{r-1}g + r - 1$, on its bottom row at positions

$$-\frac{r-1}{2}, -\frac{r-3}{2}, \ldots, \frac{r-3}{2}, \frac{r-1}{2},$$

respectively.

Proof. The proof is by induction on r. For $r = 2$, a 2-slab is just a 2-row, which is clearly balanced with downward forces of $g, 2g, g$ at $-1, 0, 1$ and upward forces of $2g+1, 2g+1$ at $-\frac{1}{2}, \frac{1}{2}$, when half of the downward force $2g$ acting at $x = 0$ is applied on the right-hand edge of the left block and the other half applied on the left-hand edge of the right block.

For the induction step, we first observe that for any $r \geq 2$ an $(r+1)$-slab can be regarded as an r-slab with an $(r+1)$-row added above and below and with an extra block added at each end of the $r-2$ rows of length $r-1$ of the r-slab. The 5-slab (shaded) contained in a 6-slab together with the added blocks is shown in Figure 16.

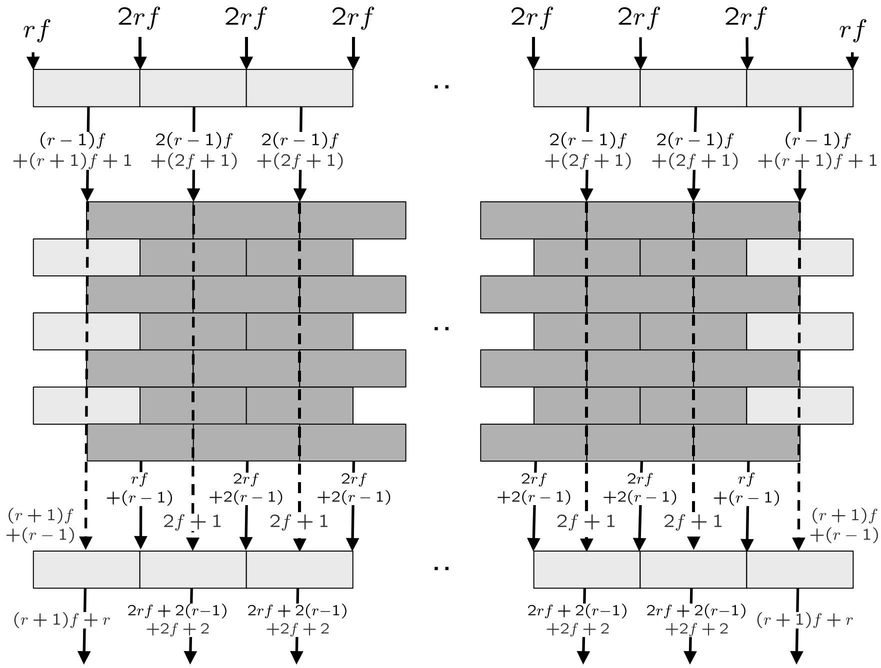

Figure 17. The proof of Lemma 4.2.

Suppose the statement of the lemma holds for r-slabs and consider an $(r+1)$-slab with the supposed forces acting on its top row. Let $f = g/r$, so that $g = rf$. As in the basis of the induction, the top row can be balanced by $r+1$ equal forces of $2rf+1$ from below (the 1 is for the weight of the blocks in the top row) acting at positions $-\frac{r}{2}, -\frac{r-2}{2}, \ldots, \frac{r-2}{2}, \frac{r}{2}$. As

$$2rf + 1 = (r-1)f + ((r+1)f + 1) = 2(r-1)f + 2f + 1,$$

we can express this constant sequence of $r+1$ forces as the sum of the following two force sequences:

$$\begin{array}{cccccc} (r-1)f, & 2(r-1)f, & 2(r-1)f, & \ldots, & 2(r-1)f, & 2(r-1)f, & (r-1)f \\ (r+1)f+1, & 2f+1, & 2f+1, & \ldots, & 2f+1, & 2f+1, & (r+1)f+1 \end{array}$$

The forces in the first sequence can be regarded as acting on the r-slab contained in the $(r+1)$-slab, which then, by the induction hypothesis, yield downward forces on the bottom row of

$$rf + r - 1, \;\; 2rf + 2(r-1), \;\; \ldots, \;\; 2rf + 2(r-1), \;\; rf + r - 1$$

at positions $-\frac{r-1}{2}, -\frac{r-3}{2}, \ldots, \frac{r-3}{2}, \frac{r-1}{2}$.

The forces of the second sequence, together with the weights of the outermost blocks of the $(r+1)$-rows, are passed straight down through the rigid structure of the r-slab to the bottom row. The combined forces acting down on the bottom row are now

$$(r+1)f+r-1,\ rf+r-1,\ 2f+1,\ 2rf+2(r-1),\ 2f+1,\\ \dots,\ 2f+1,\ rf+r-1,\ (r+1)f+r-1$$

at positions $-\frac{r}{2}, -\frac{r-1}{2}, \dots, \frac{r-1}{2}, \frac{r}{2}$. The bottom row is in equilibrium when the sequence of upward forces

$$(r+1)f+r,\quad 2(r+1)f+2r,\quad 2(r+1)f+2r,\\ \dots,\quad 2(r+1)f+2r,\quad 2(r+1)f+2r,\quad (r+1)f+r$$

is applied on the bottom row at positions $-\frac{r}{2}, -\frac{r-2}{2}, \dots, \frac{r-2}{2}, \frac{r}{2}$, as required. ∎

Theorem 4.3. *For any $d \geq 2$, a parabolic d-stack is balanced, contains*

$$\frac{d(d-1)(2d-1)}{3}+1$$

blocks, and has an overhang of $\frac{d}{2}$.

Proof. The balance of a parabolic d-stack follows by a repeated application of Lemma 4.2. For $2 \leq r \leq d$, let $g(r)$ denote the value of g in Lemma 4.2 for the r-slab in the d-stack. Although the argument does not rely on the specific values that $g(r)$ assumes, it can be verified that $g(r) = \frac{1}{r}\sum_{i=r}^{d-1} i^2$. Note that $g(d) = 0$, as no downward forces are exerted on the top row of the d-slab, which is also the top row of the d-stack, and that $g(r-1) = \frac{r}{r-1}g(r) + r - 1$, as required by Lemma 4.2. ∎

Theorem 4.4. $D(n) \geq (\frac{3n}{16})^{1/3} - \frac{1}{4}$ *for all n.*

Proof. Choose d so that

$$\frac{(d-1)d(2d-1)}{3}+1 \leq n \leq \frac{d(d+1)(2d+1)}{3}.$$

Then Theorem 4.3 shows that a d-stack yields an overhang of $d/2$ and can be constructed using n or fewer blocks. Any extra blocks can be just placed in a vertical pile in the center on top of the stack without disturbing balance (or arbitrarily scattered on the table). Hence

$$n < \frac{2(d+\frac{1}{2})^3}{3} \quad \text{and so} \quad D(n) \geq d/2 > \left(\frac{3n}{16}\right)^{1/3} - \frac{1}{4}.$$ ∎

In Section 3 we claimed that optimal stacks are spinal *only* for $n \leq 19$. We can justify this claim for $n \leq 30$ by exhaustive search, while comparison of the lower bound from Theorem 4.4 with the upper bound of $S(n) < 1 + \ln n$ from Corollary 3.7 deals with the range $n \geq 5000$. The intermediate values of n can be covered by combining a few explicit constructions, such as the stack shown in Figure 20, with numerical bounds using Lemma 3.5.

Can parabolic d-stacks be built incrementally by laying one brick at a time? The answer is no, as the bottom three rows of a parabolic stack form an unbalanced inverted 3-triangle. The inverted 3-triangle remains unbalanced when the first block of the fourth row is laid down. Furthermore, the bottom six rows, on their own, are also not balanced. These, however, are the only obstacles to an incremental row-by-row and block-by-block construction of parabolic stacks and they can be overcome by the modified parabolic stacks shown in Figure 18. We simply omit the lowest block and move the whole stack half a block length to the left. The bricks can now be laid row by row, going in each row from the center outward, alternating between the left and right sides, with the left side, which is over the table, taking precedence. The numbers in Figure 18 indicate the order in which the blocks are laid. Thus, unlike with harmonic stacks, it is possible to construct an arbitrarily large overhang using sufficiently many blocks, *without* knowing the desired overhang in advance.

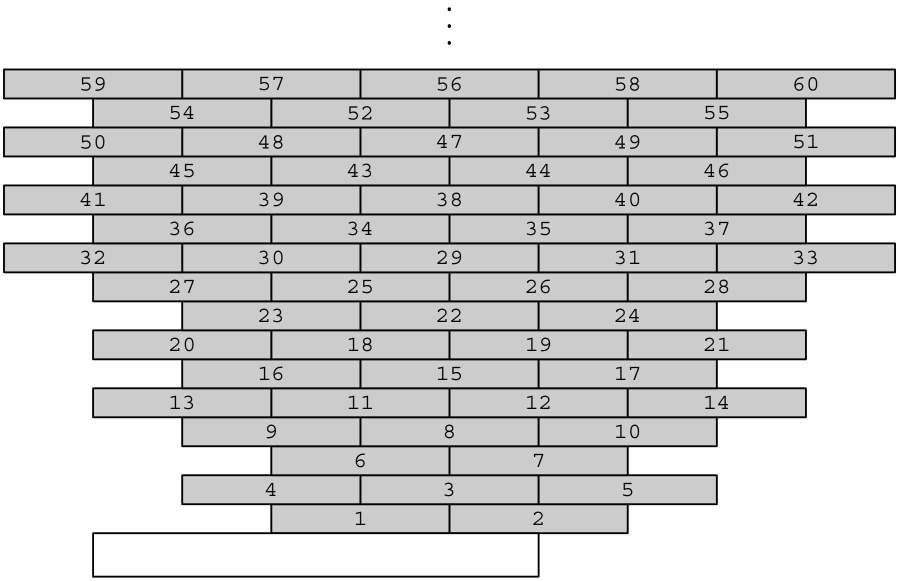

Figure 18. Incremental block-by-block construction of modified parabolic stacks.

5. GENERAL STACKS. We saw in Section 2 that the problem of checking whether a given stack is balanced reduces to checking the feasibility of a system of linear equations and inequalities. Similarly, the minimum total weight of the point weights that are needed to balance a given loaded stack can be found by solving a linear program.

Finding a stack with a given number of blocks, or a loaded stack with a given total weight, that achieves maximum overhang seems, however, to be a much harder computational task. To do so, one should, at least in principle, consider all possible combinatorial stack structures and for each of them find an optimal placement of the blocks. The *combinatorial structure* of a stack specifies the contacts between the blocks of the stack, i.e., which blocks rest on which, and in what order (from left to right), and which rest on the table.

The problem of finding a (loaded) stack with a given combinatorial structure with maximum overhang is again not an easy problem. As both the forces and their locations are now unknowns, the problem is not linear, but rather a constrained quadratic programming problem. Though there is no general algorithm for efficiently finding the global optimum of such constrained quadratic programs, such problems can still be solved in practice using nonlinear optimization techniques.

For stacks with a small number of blocks, we enumerated all possible combinatorial stack structures and numerically optimized each of them. For larger numbers of blocks this approach is clearly not feasible and we had to use various heuristics to cut down the number of combinatorial structures considered. The stacks of Figures 3, 4, 5, and 6 were found using extensive numerical experimentation. The stacks of Figures 3, 4, and 5 are optimal, while the stacks of Figure 6 are either optimal or very close to being so.

The collections of forces that balance the loaded stacks of Figure 6 (and the loaded stacks contained in the stacks of Figures 3, 4, and 5) have the following interesting properties. First, the balancing collections of forces of these stacks are unique. Second, almost all downward forces in these collections are applied at the edges of blocks. The only exceptions occur when a downward force is applied on a *right-protruding* block, i.e., a rightmost block in a level that protrudes beyond the rightmost block of the level above it. In addition, all point weights are placed on the left-hand edges of *left-protruding* blocks, where left-protruding blocks are defined in an analogous way. The table, of course, supports the (only) block that rests on it at its right-hand edge. A collection of balancing forces that satisfies these conditions is said to be *well-behaved*. A schematic description of well-behaved collections of balancing forces is given in Figure 19. The two right-protruding blocks are shown with a slightly lighter shading. A right-protruding block is always adjacent to the block on its left. We conjecture that forces that balance optimal loaded stacks are always well-behaved.

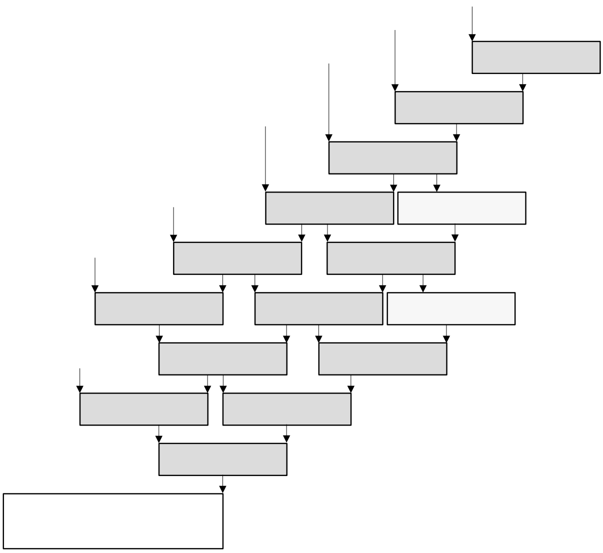

Figure 19. A schematic description of a well-behaved set of balancing forces.

A useful property of well-behaved collections of balancing forces is that the total weight of the stack and the positions of its blocks uniquely determine all the forces in the collection. This follows from the fact that each block has either two downward forces acting upon it at specified positions, namely at its two edges, or just a single force in an unspecified position. Given the upward forces acting on a block, the downward force or forces acting upon it can be obtained by solving the force and moment equations of the block. All the forces in the collection can therefore be determined in a bottom-up fashion. We conducted most of our experiments, on blocks with more than 30 blocks, on loaded stacks balanced by well-behaved sets of balancing forces.

We saw in Section 2 that loaded stacks of total weight 3, 5, and 7 achieve a larger overhang than the corresponding unloaded stacks, simply because the number of blocks available for use in their balancing sets is smaller than the number of point weights to be applied. The loaded stacks of Figure 6 exhibit another trivial impediment to the conversion of loaded stacks into standard ones: the point weight to be applied in the lowest position has magnitude less than 1. Thus, these stacks can be converted into standard ones only after making some small adjustments. These adjustments have only a very small effect on the overhang achieved. Thus, although we believe that the difference between the maximum overhangs achieved by loaded and unloaded stacks is bounded by a small universal constant, we also believe that for most sizes, loaded stacks yield slightly larger overhangs.

Although the placements of the blocks in the optimal, or close to optimal, stacks of Figure 6 are somewhat irregular, with some small (essential) gaps between blocks of the same layer, at a high level, these stacks seem to resemble brick-wall stacks, as defined in Section 4. This, and the fact that brick-wall stacks were used to obtain the $\Omega(n^{1/3})$ lower bound on the maximum overhang, indicate that it might be interesting to investigate the maximum overhang that can be achieved using brick-wall stacks.

The parabolic brick-wall stacks of Section 4 were designed to enable a simple inductive proof of their balance. Parabolic stacks, however, are far from being optimal brick-wall stacks. The balanced 95-block symmetric brick-wall stack with an overhang of 4 depicted in Figure 20, for example, contains fewer blocks and achieves a larger overhang than that achieved by the 111-block overhang-3 parabolic stack of Figure 9.

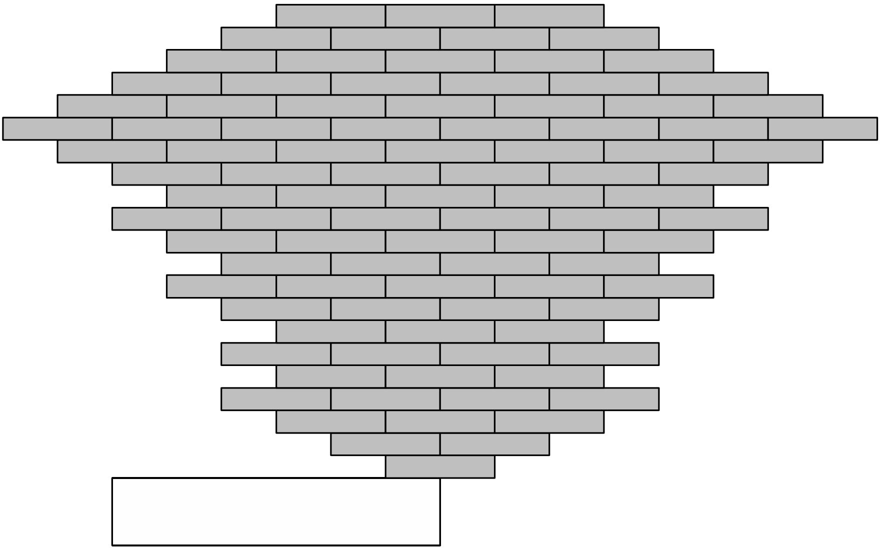

Figure 20. A 95-block symmetric brick-wall stack with overhang 4.

Loaded brick-wall stacks are especially easy to experiment with. Empirically, we have again discovered that the minimum weight collections of forces that balance them turn out to be well-behaved, in the formal sense defined above. When the brick-wall stacks are *symmetric* with respect to the $x = 0$ axis, and have a flat top, point weights are attached only to blocks at the top layer of the stack. Protruding blocks, both on the left and on the right, then simply serve as *props*, while all other blocks are perfect *splitters*, i.e., they are supported at the center of their lower edge and they support other blocks at the two ends of their upper edge. In non-symmetric brick-wall stacks it is usually profitable to use the left-protruding blocks as splitters and not as props, attaching point weights to their left ends. A schematic description of well-behaved forces that balance symmetric and asymmetric brick-wall stacks is shown in Figure 21. As can be seen, all forces in such well-behaved collections are linear functions of w, the total weight of the stack. This allows us, in particular, to find the minimum total weight needed to balance a brick-wall loaded stack *without* solving a linear program. We simply choose the smallest total weight w for which all forces are nonnegative. This observation enabled us to experiment with huge symmetric and asymmetric brick-wall stacks.

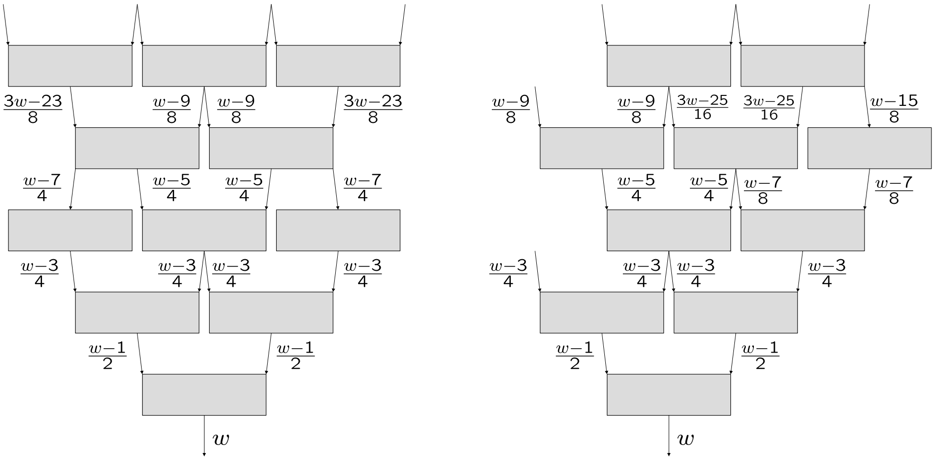

Figure 21. A schematic description of well-behaved collections of forces that balance symmetric and asymmetric brick-wall stacks.

The best symmetric loaded brick-wall stacks with overhangs 10 and 50 that we have found are shown in Figures 22 and 23. Their total weights are about 1151.76 and 115,467, respectively. The blocks in the larger stack are so small that they are not shown individually. We again believe that these stacks are close to being the optimal stacks of their kind. They were found using a local search approach. In particular, these stacks cannot be improved by widening or narrowing layers, or by adding or removing single layers. Essentially the same symmetric stacks were obtained by starting from almost any initial stack and repeatedly improving it by widening, narrowing, adding, and removing layers.

As can be seen from Figures 22 and 23, the shapes of optimal symmetric loaded stacks, after suitable scaling, seem to tend to a limiting curve. This curve, which we have termed the *vase*, is similar to but different from that of an inverted normal distribution. We have as yet no conjecture for its equation.

We have conducted similar experiments with asymmetric loaded brick-wall stacks. The best such stack with overhang 10 that we have found is shown in Figure 24. Its total weight of about 1128.84 is about 3.38% less than the weight of the symmetric stack of Figure 22. The scaled shapes of optimal asymmetric loaded brick-wall stacks seem again to tend to a limiting curve which we have termed the *oil lamp*. We again have no conjecture for its equation.

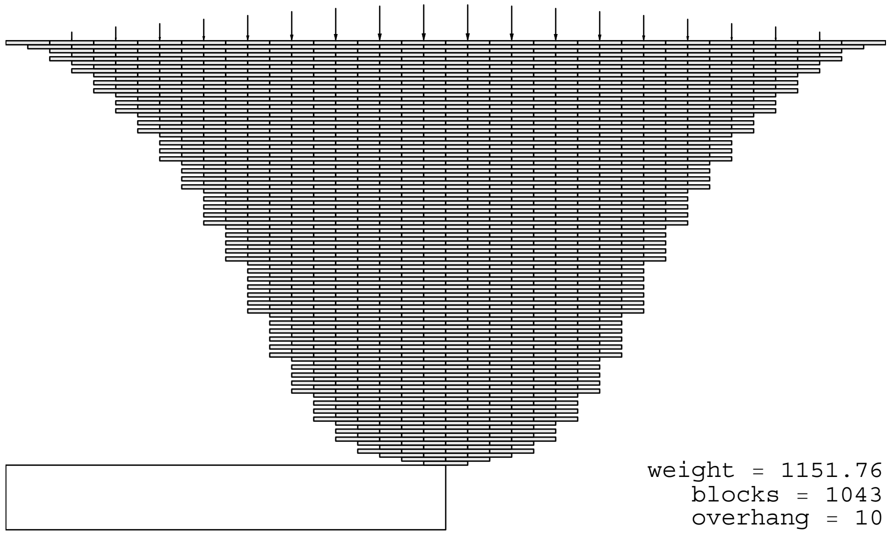

Figure 22. A symmetric loaded brick-wall stack with an overhang of 10.

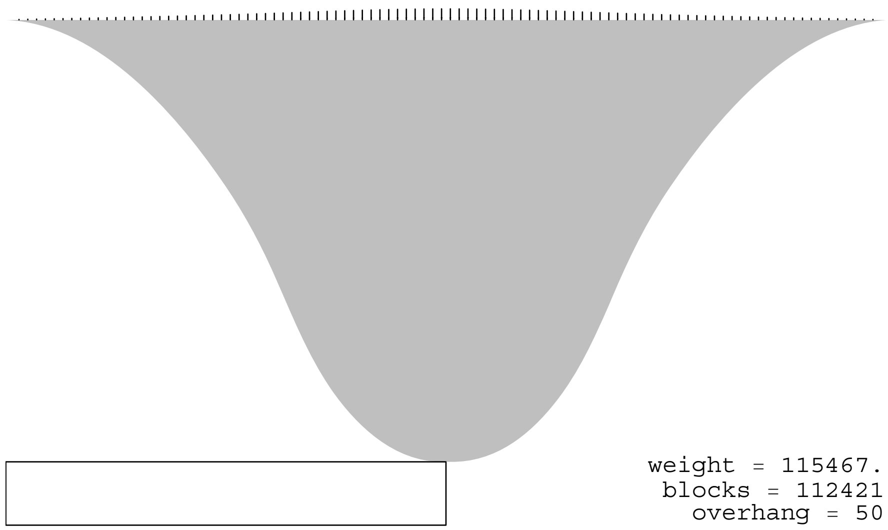

Figure 23. A scaled outline of a loaded brick-wall stack with an overhang of 50.

6. OPEN PROBLEMS.

Some intriguing problems still remain open. In a subsequent paper [**15**], we show that the $\Omega(n^{1/3})$ overhang lower bound presented here is optimal, up to a constant factor, but it would be interesting to determine the largest constant c_{over} for which overhangs of $(c_{\text{over}} - o(1))n^{1/3}$ are possible. Can this constant c_{over} be achieved using stacks that are simple to describe, e.g., brick-wall stacks, or simple modifications of them, such as brick-wall stacks with adjacent levels having a displacement other than $\frac{1}{2}$, or small gaps left between the blocks of the same level?

What are the limiting vase and oil lamp curves? Do they yield, asymptotically, the maximum overhangs achievable using symmetric and general stacks?

Another open problem is the relation between the maximum overhangs achievable using loaded and unloaded stacks. We believe, as expressed in Conjecture 2.7, that

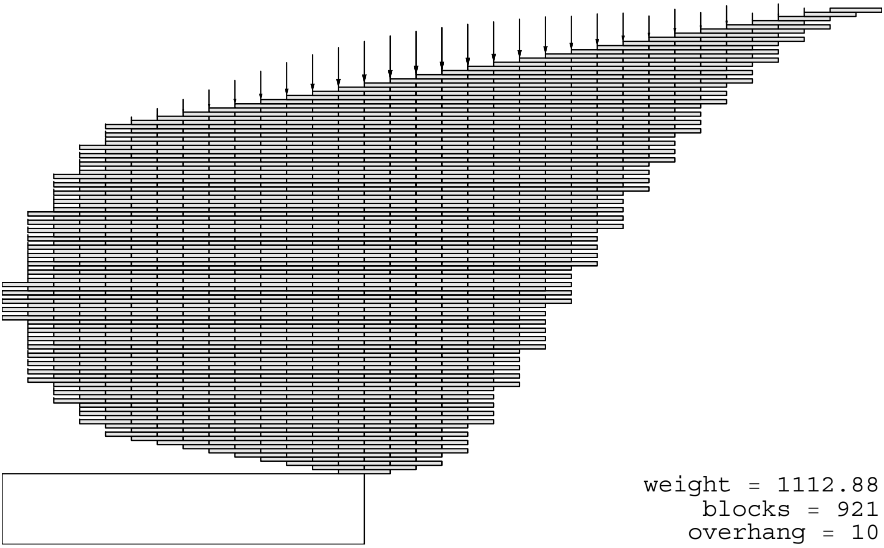

Figure 24. An asymmetric loaded brick-wall stack with an overhang of 10.

the difference between these two quantities tends to 0 as the size of the stacks tends to infinity. We also conjecture that $D^*(n) - D(n) \leq D^*(3) - D(3) = \frac{5-2\sqrt{6}}{6} \simeq 0.017$, for every $n \geq 1$.

Our notion of balance, as defined formally in Section 2, allows stacks to be precarious: stacks that achieve maximum overhang are always on the verge of collapse. It is not difficult, however, to define more robust notions of balance, where there is some *stability*. In one such natural definition, a stack is *stable* if there is a balancing set of forces in which none of the forces acts at the edge of any block.

We note in passing that Farkas' lemma, or the theory of linear programming duality (see [**17**]), can be used to derive an equivalent definition of stability: a stack is stable if and only if every feasible infinitesimal motion of the blocks of the stack increases the total potential energy of the system.

This requirement of stability raises some technical difficulties but does not substantially change the nature of the overhang problem. Our parabolic d-stacks, for example, can be made stable by adding a $(d-1)$-row symmetrically placed on top. The proof of this is straightforward but not trivial. We believe that for any $n \neq 3$, the loss in the overhang due to this stricter definition is infinitesimal.

Our analysis of the overhang problem was made under the *no friction* assumption. All the forces considered were therefore vertical. The presence of friction introduces horizontal forces and thus changes the picture completely, as also observed by Hall [**9**]. We can show that there is a fixed coefficient of friction such that the inverted triangles are all balanced, and so achieve overhang of order $n^{1/2}$.

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