The Arithmetic–Geometric Mean Inequality and the Constant e

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T. N. T. Goodman [**1**] and C.W. Barnes [**2**] gave two interesting proofs of the limit $\lim_{n\to\infty}(1+1/n)^n = e$ using the inequalities

$$\frac{e}{1+1/n} \le \left(1+\frac{1}{n}\right)^n \le e. \tag{1}$$

(See also [**3**, p. 354].)

In this note we present a very elementary proof that the inequalities

$$\left(1+\frac{1}{n}\right)^n < e \le \left(1+\frac{1}{m-1}\right)^m \tag{2}$$

hold for every integers $n > 0$ and $m > 1$. We use only the well-known *arithmetic–geometric mean inequality* (AGMI): For any n positive real numbers $x_1, x_2, \ldots, x_n$, we have

$$\sqrt[n]{x_1 x_2 \ldots x_n} \le \frac{x_1 + x_2 + \cdots + x_n}{n},$$

or, equivalently,

$$x_1 x_2 \ldots x_n \le \left(\frac{x_1 + x_2 + \cdots + x_n}{n}\right)^n; \tag{3}$$

equality holds if and only if $x_1 = x_2 = \cdots = x_n$. (See, e.g., [**4**] for more on the AGMI.)

Now set $x_n = (1+1/n)^n$. By the AGMI (3) with $n+1$ terms, we have

$$\begin{aligned} x_n = \left(1+\frac{1}{n}\right)^n \cdot 1 &< \left[\frac{\overbrace{\left(1+\tfrac{1}{n}\right)+\cdots+\left(1+\tfrac{1}{n}\right)}^{n}+1}{n+1}\right]^{n+1} \\ &= \left(\frac{n+1+1}{n+1}\right)^{n+1} = x_{n+1}, \end{aligned}$$

which proves that the sequence $\{x_n\}$ is increasing.

For an arbitrary positive integer $q > 1$, let the equation $1/p + 1/q = 1$ determine the number $p > 1$; note that $1 + q/p = q$. Using (3) again, we see that

$$x_n \cdot \left(\frac{1}{p}\right)^q = \left(1+\frac{1}{n}\right)^n \left(\frac{1}{p}\right)^q \le \left[\frac{\overbrace{\left(1+\frac{1}{n}\right)+\cdots+\left(1+\frac{1}{n}\right)}^{n}+\overbrace{\frac{1}{p}+\cdots+\frac{1}{p}}^{q}}{n+q}\right]^{n+q}$$
$$= \left(\frac{n+1+\frac{q}{p}}{n+q}\right)^{n+q} = 1,$$

which implies that

$$x_n = \left(1+\frac{1}{n}\right)^n \le p^q. \tag{4}$$

Thus the sequence $\{x_n\}$ is increasing and bounded, and so converges to a limit we call e. By (4), e satisfies

$$\left(1+\frac{1}{n}\right)^n < e \le p^q.$$

If we set $q = m > 1$, then $p = m/(m-1)$, and we obtain the inequalities (2).

The special case $m = n+1$ in (2) gives the inequalities

$$\left(1+\frac{1}{n}\right)^n < e \le \left(1+\frac{1}{n}\right)^{n+1} := y_n.$$

Next, we show how to apply the AGMI again to prove the *strict* inequality

$$x_n = \left(1+\frac{1}{n}\right)^n < e < \left(1+\frac{1}{n}\right)^{n+1} = y_n, \tag{5}$$

and that $\lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n$. Note that inequality (5) is stronger than (1). In fact,

$$\frac{y_{n+1}}{y_n} = \left(1+\frac{1}{n+1}\right)^{n+1} \left(\frac{1+\frac{1}{n+1}}{\left(1+\frac{1}{n}\right)^{n+1}}\right) = \left(\frac{n(n+2)}{(n+1)^2}\right)^{n+1} \left(1+\frac{1}{n+1}\right)$$
$$= \left(1-\frac{1}{(n+1)^2}\right)^{n+1} \left(1+\frac{1}{n+1}\right).$$

Using the AGMI again, we have

$$\frac{y_{n+1}}{y_n} = \left(1-\frac{1}{(n+1)^2}\right)^{n+1} \left(1+\frac{1}{n+1}\right)$$
$$< \left[\frac{\overbrace{1-\frac{1}{(n+1)^2}+\cdots+1-\frac{1}{(n+1)^2}}^{n+1}+1+\frac{1}{n+1}}{n+2}\right]^{n+2} = 1.$$

Thus the sequence $\{y_n\}$ is decreasing and obviously positive, so it converges to a limit we call e, with $y_n > e$. On the other hand,

$$\frac{x_n}{x_{n+1}} = \frac{\left(1+\frac{1}{n}\right)^n}{\left(1+\frac{1}{n+1}\right)^{n+1}} = \left[\frac{(n+1)^2}{n(n+2)}\right]^n \cdot \frac{n+1}{n+2}$$

$$< \left[\frac{\overbrace{\frac{(n+1)^2}{n(n+2)} + \cdots + \frac{(n+1)^2}{n(n+2)}}^{n} + \frac{n+1}{n+2}}{n+1}\right]^{n+1}$$

$$= \left(\frac{\frac{(n+1)^2}{n+2} + \frac{n+1}{n+2}}{n+1}\right)^{n+1} = 1.$$

This shows that the sequence $\{x_n\}$ is increasing, and clearly $x_n < y_n < y_1 = 4$. Hence $\{x_n\}$ converges, and $x_n < \lim_{n\to\infty} x_n$ for all n. We can prove easily that $\lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n = e$, and (5) follows.

REFERENCES

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3. Chung-Lie Wang, Simple inequalities and old limits, *Amer. Math. Monthly* **96** (1989), 354–355.
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50 Years Ago in the MAGAZINE

From John Lowe's article, "Automatic Computation as an Aid in Aeronautical Engineering," Vol. **25**, No. 1, (Sept.–Oct., 1951):

A tremendous amount of numerical labor is involved in designing today's aircraft and missiles.... For example, in one phase of the fuselage stress analysis of a single configuration of the DC-6 airplane, 200,000 multiplications and additions were performed, and the flutter analysis required 1,000,000 multiplications and additions. In these facts we find our first reason for the use of automatic computers....

Machines must be programmed in the most minute detail. Problems which one does not consider in computing with a desk calculator become of paramount importance. For example, the determination of whether or not a given quantity is zero may require some planning. If data in graphical form are to be introduced into a computation, these data must be translated to numerical form, perhaps by some curve fitting method....

There exists a stringent shortage of people qualified for this work and this shortage shows every sign of becoming more acute. Capable people are being paid well. The field is so new that few people even know it exists. So little is known that ambitious people can be doing truly original work early in their careers. I hope that it will receive increasing recognition in school curricula and from student counselors. Today, few other fields offer technical or scientific college graduates the opportunity for advancement that is offered by computing.