Fibonacci Identities via the Determinant Sum Property

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There is a long tradition of using matrices and determinants to study Fibonacci numbers. For example, Bicknell-Johnson and Spears [**3**] use elementary matrix operations and determinants to generate classes of identities for generalized Fibonacci numbers, and Cahill and Narayan [**4**] show how Fibonacci and Lucas numbers arise as determinants of some tridiagonal matrices. Another recent paper, by Benjamin, Cameron, and Quinn [**2**], provides combinatorial interpretations for Fibonacci identities using determinants. Koshy's extensive survey text [**7**] of the Fibonacci numbers includes two chapters on the use of matrices and determinants as well.

However, none of this work utilizes the sum property for determinants [**6**]:

> *If A, B, and C are matrices with identical entries except that one row (column) of C, say the kth, is the sum of the kth rows (columns) of A and B, then $|A| + |B| = |C|$.*

In this note we use this property to give a new proof of the following identity for the Fibonacci numbers [**1**]:

$$F_m F_n - F_{m-r} F_{n+r} = (-1)^{m-r} F_r F_{n+r-m}. \tag{1}$$

The following identities [**8**] are special cases of (1):

Cassini's identity:	$F_{n+1} F_{n-1} - F_n^2 = (-1)^n$	(2)
d'Ocagne's identity:	$F_{m+1} F_n - F_m F_{n+1} = (-1)^m F_{n-m}$	(3)
Catalan's identity:	$F_n^2 - F_{n+r} F_{n-r} = (-1)^{n-r} F_r^2$	(4)

Our proof of (1) has three stages; we reach Cassini's identity at the end of the first stage and d'Ocagne's at the end of the second.

Generalized Fibonacci sequences

The Fibonacci sequence is defined by the initial values $F_0 = 0$ and $F_1 = 1$ and the recurrence relation $F_n = F_{n-1} + F_{n-2}$, where $n \geq 2$. A generalized Fibonacci sequence is a sequence of numbers $\{G_n\}$ that obeys the same Fibonacci recurrence relation but with arbitrary starting values. Thus $G_n = G_{n-1} + G_{n-2}$, while G_0 and G_1 can be any numbers. It can readily be shown ([**5**], p. 493), that

$$G_n = G_0 F_{n-1} + G_1 F_n. \tag{5}$$

First stage of the proof

We begin with the 2×2 identity matrix:

$$A_0 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

For each $n \geq 1$, we now recursively construct A_n from A_{n-1} by adding the second row of A_{n-1} to the first and then interchanging the two rows.

Then, for example,

$$A_1 = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}, \; A_2 = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}, \; A_3 = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}, \text{ and } A_4 = \begin{bmatrix} 2 & 3 \\ 3 & 5 \end{bmatrix}.$$

The entries in each A_n appear to be Fibonacci numbers, and, in fact, it is easy to show by induction that

$$A_n = \begin{bmatrix} F_{n-1} & F_n \\ F_n & F_{n+1} \end{bmatrix}.$$

The two steps involved in generating A_n from A_{n-1} are both elementary matrix row operations: In the first, one row is added to another, while in the second, two rows are swapped. The first of these does not affect the determinant, and the second changes only the sign. Since $|A_0| = 1$, it follows that $|A_n| = (-1)^n$. Since

$$A_n = \begin{bmatrix} F_{n-1} & F_n \\ F_n & F_{n+1} \end{bmatrix},$$

the definition of the determinant implies that $F_{n+1}F_{n-1} - F_n^2 = (-1)^n$, Cassini's identity (2). (We note that there is another proof of Cassini's identity that uses determinants; see, for example, Knuth [**5**], p. 81.)

Second stage

The second stage brings us to d'Ocagne's identity. Let B_0 be the matrix

$$\begin{bmatrix} F_n & F_n \\ F_{n+1} & F_{n+1} \end{bmatrix}.$$

Construct B_1 by adding column 1 of A_n to that of B_0. Thus

$$B_1 = \begin{bmatrix} F_{n+1} & F_n \\ F_{n+2} & F_{n+1} \end{bmatrix},$$

and by the determinant sum property, $|B_1| = (-1)^n$. For $r \geq 2$, we now construct B_r by adding the first column of B_{r-2} to that of B_{r-1}. Thus,

$$B_2 = \begin{bmatrix} F_{n+2} & F_n \\ F_{n+3} & F_{n+1} \end{bmatrix}, \; B_3 = \begin{bmatrix} F_{n+3} & F_n \\ F_{n+4} & F_{n+1} \end{bmatrix}, \; B_4 = \begin{bmatrix} F_{n+4} & F_n \\ F_{n+5} & F_{n+1} \end{bmatrix},$$

and so forth.

An easy induction proof shows that

$$B_r = \begin{bmatrix} F_{n+r} & F_n \\ F_{n+r+1} & F_{n+1} \end{bmatrix}. \tag{6}$$

From the determinant sum property, we have $|B_r| = |B_{r-1}| + |B_{r-2}|$. This recurrence relation implies that the sequence $\{|B_r|\}$ is actually a generalized Fibonacci sequence. Since $|B_0| = 0$ and $|B_1| = (-1)^n$, it follows from (5) that for $r \geq 0$,

$$|B_r| = (-1)^n F_r. \tag{7}$$

Since $|B_0| = 0$, not only is the sequence $\{|B_r|\}$ a generalized Fibonacci sequence, it is actually the Fibonacci sequence or its negative.

By (6) and the definition of the determinant, $|B_r| = F_{n+r}F_{n+1} - F_n F_{n+r+1}$. Let $m = n + r$, and it follows from this result and (7) that $F_{n+1}F_m - F_n F_{m+1} = (-1)^n F_{m-n}$, d'Ocagne's identity (3). Consequently, we can see that the Fibonacci numbers on the right-hand side of d'Ocagne's identity arise naturally as a result of adding determinants.

Third stage

In the final stage of the proof we again generate a sequence of matrices, but instead of adding the first columns of the two previous matrices together to obtain the next matrix in the sequence, we add the bottom rows. Let

$$C_0 = \begin{bmatrix} F_n & F_{n-r} \\ F_n & F_{n-r} \end{bmatrix} \quad \text{and let} \quad C_1 = \begin{bmatrix} F_n & F_{n-r} \\ F_{n+1} & F_{n-r+1} \end{bmatrix},$$

the matrix obtained by replacing n with $n - r$ in B_r. Now, for $s \geq 2$, construct C_s by adding the second row of C_{s-2} to that of C_{s-1}.

Then

$$C_2 = \begin{bmatrix} F_n & F_{n-r} \\ F_{n+2} & F_{n-r+2} \end{bmatrix}, \; C_3 = \begin{bmatrix} F_n & F_{n-r} \\ F_{n+3} & F_{n-r+3} \end{bmatrix}, C_4 = \begin{bmatrix} F_n & F_{n-r} \\ F_{n+4} & F_{n-r+4} \end{bmatrix},$$

and so forth. Again, we can show by induction that

$$C_s = \begin{bmatrix} F_n & F_{n-r} \\ F_{n+s} & F_{n-r+s} \end{bmatrix}. \tag{8}$$

As in the second stage of the proof, it follows from the determinant sum property that $|C_s| = |C_{s-1}| + |C_{s-2}|$. This means that the sequence $\{|C_s|\}$ is also a generalized Fibonacci sequence. By (7), $|C_1| = (-1)^{n-r}F_r$. Since $|C_0| = 0$, it then follows from (5) that for $s \geq 0$,

$$|C_s| = [(-1)^{n-r}F_r]\,F_s. \tag{9}$$

Like the determinants $|B_r|$, the determinants $|C_s|$ form a multiple of the Fibonacci sequence.

By (8) and the definition of the determinant, $|C_s| = F_n F_{n-r+s} - F_{n+s} F_{n-r}$. Letting $s = m - n + r$, we see from this result and (9) that $F_n F_m - F_{m+r} F_{n-r} = (-1)^{n-r} F_r F_{m-n+r}$, which is (1). Catalan's identity (4) is the special case $m = n$. As in Stage 2 and the proof of d'Ocagne's identity, the Fibonacci numbers on the right-hand side of (1) arise naturally as a result of the generalized Fibonacci sequence of determinants obtained by the repeated use of the determinant sum property.

Acknowledgement. The author would like to thank an anonymous referee for many suggestions that improved the exposition in this paper.

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Names in Boxes Puzzle

C. The Solution (Puzzle on page 260; strategy on page 285) The random assignment of a box with a name in it to each prisoner is just a permutation of the one hundred names, chosen at random from among the set of all such permutations. In inspecting boxes, each prisoner is following a cycle of that permutation, and if they don't exceed the fifty-box limit, they succeed in finding their own name. Thus, if the permutation *has no cycle of length greater than fifty*, this process will work and the prisoners will be spared.

In fact, the probability that a random permutation of $2n$ objects has no cycle of length greater than n is at least $1 - \ln 2$, or about 30.68%. To see this, we let $k > n$ and count the permutations having a cycle of length exactly k. Using routine counting techniques, we find that this number is $n!/k$. Since there can be at most one k-cycle in a given permutation, the probability that there is one is exactly $1/k$. It follows that the probability that there is no long cycle is

$$1 - \frac{1}{n+1} - \frac{1}{n+2} - \cdots - \frac{2}{2n} = 1 - H_{2n} + H_n,$$

where H_m is the sum of the reciprocals of the first m positive integers, approximately $\ln m$. Thus our probability is about $1 - \ln 2$, and in fact is always a bit larger. For $n = 50$, we find that the prisoners survive with probability of about 31.18%.

Editor's Note. Peter Winkler reports that this puzzle originated with the Danish computer scientist Peter Bro Miltersen. It will appear in Volume II of Winkler's *Mathematical Puzzles: A Connoisseur's Collection* (A.K. Peters Ltd.).