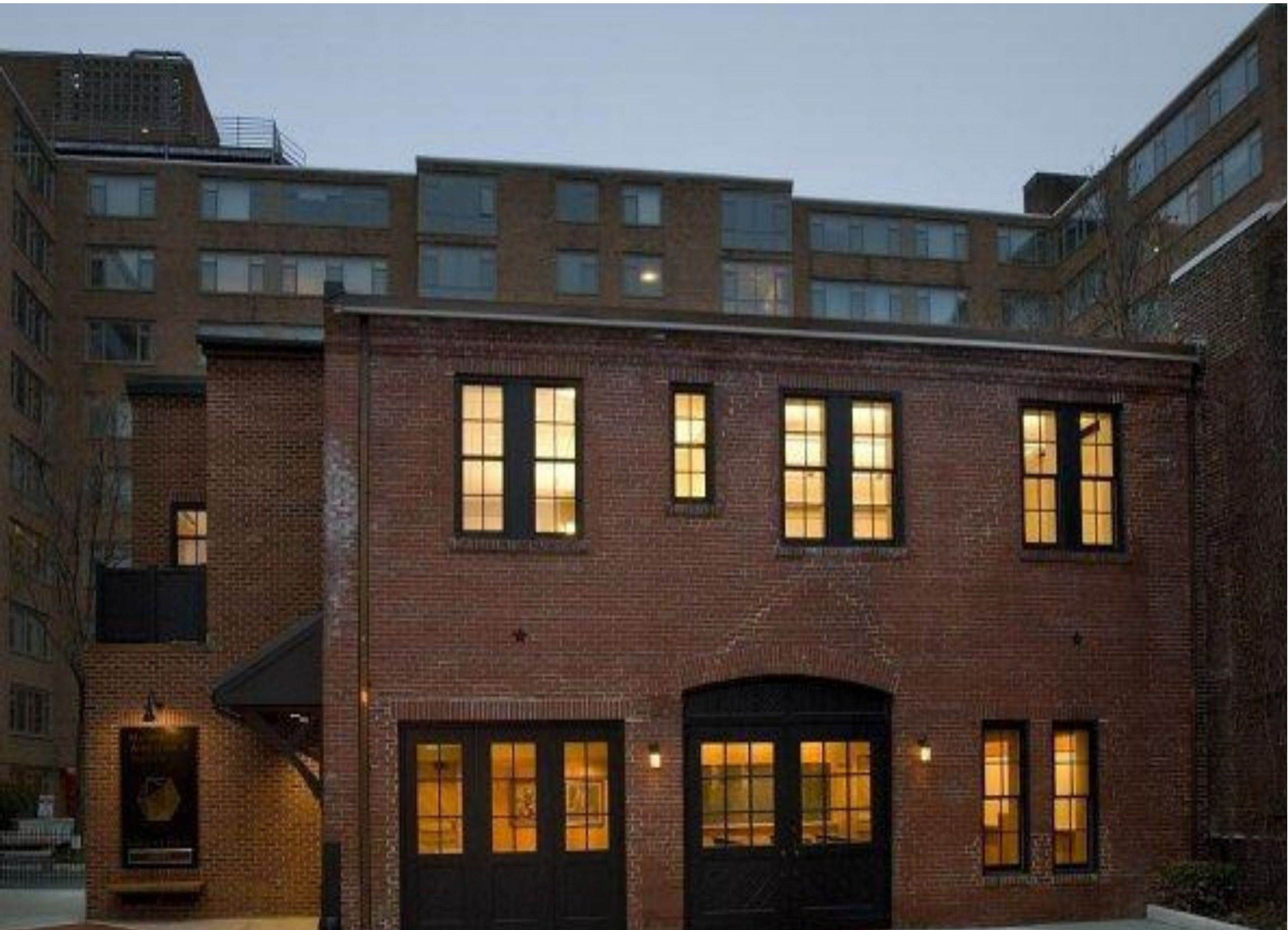

Linear Algebra

A proposed update for the 2015 CUPM Guide to Undergraduate Programs in Mathematics Sciences

July 29, 2022

MATHEMATICAL ASSOCIATION OF AMERICA

Linear Algebra

Author Information

Eugene Boman, Pennsylvania State University
Sheldon Axler, San Francisco State University
Robert A. Beezer, University of Puget Sound
Minerva Catral, Xavier University
Judith McDonald, Washington State University
Sepideh Stewart, University of Oklahoma
David Strong, Pepperdine University
Oscar Vega, California State University, Fresno
Megan Wawro, Virginia Tech

Overview

Many areas of mathematics rely on fundamental results from linear algebra. Moreover, the concepts and techniques of linear algebra underlie much of modern science, engineering, and technology. Thus it is imperative that this most fundamental of courses be well represented in the curriculum of every Science, Technology, Engineering, and Mathematics (STEM) major. Because of its extreme versatility there are many applications that can be used to illuminate key concepts. Moreover, the ubiquity of computing technology means that routine computations, once their need is understood, can be demoted in favor of higher-level abstractions.

This committee is well aware that there are many ways in which the topics of a first course in linear algebra might be organized. Naturally each department and faculty member should organize the material in the manner that works best for them, their students, and their programs. Hence, we have refrained from making specific organizational recommendations in the form of a sample syllabus. Instead, we have opted to codify the topics we believe should be the focus of a first course.

Our recommendations have been particularly informed and guided by the recommendations in [Carlson] and [Stewart]. We quote from the latter (p. 814):

> The new recommendations suggest teaching linear algebra sooner in the curriculum, removing calculus as a prerequisite, considering the needs of industry, being aware of the latest research in linear algebra education, taking advantage of technology in teaching, motivating concepts with applications, and developing second courses in linear algebra.

Student Audience

At most colleges and universities, a first course in linear algebra is required for students majoring in mathematics, computer science, or engineering. Students majoring in other sciences, as well as students majoring in business or economics, could also benefit from a first course in linear algebra.

A second course in linear algebra usually attracts mostly mathematics majors. However, because of the increasing importance of linear algebra in business and industry, some students in other majors also enroll in a second course in linear algebra.

Cognitive Goals Addressed

Students develop effective thinking and communication skills.
A first course in linear algebra should contain both abstraction and concrete computation to help with the development of mathematical maturity.

The process of mapping a specific problem onto the theoretical concepts helps learners develop clear and effective thinking by forcing them to view the problem through the lens of theory. Explaining the solution process promotes effective communication skills.

Students learn to link applications and theory.
The applications of linear algebra are manifold. We quote from the 2015 edition of this Guide:

> Among these, in no particular order, are Markov chains, graph theory, correlation coefficients, cryptology, interpolation, long-term weather prediction, the Fibonacci sequence, difference equations, systems of linear differential equations, network analysis, linear least squares, graph theory, Leslie population models, the power method of approximating the dominant eigenvalue, linear programming, computer graphics, coding theory, spectral decomposition, principal component analysis, discrete and continuous dynamical systems, iterative solutions of linear systems, image processing, and traffic flow.

With the explosion of techniques used in data analysis and artificial intelligence, the list of applications is expanding and changing at a rapid rate.

Modern applications can and should be used to introduce and motivate specific concepts and display how a given concept interacts, and intersects, with others. For example, a discussion of Google's page rank algorithm could be used to motivate least squares solutions from which a segue into orthogonality, singular vectors and the Singular Value Decomposition (SVD) of a matrix, and eigenvalues and the Spectral Decomposition could follow naturally. Much the same observation can be made of nearly any of the applications listed above. The link between the abstract theory of linear algebra and its applications is deep and profound. It can and should be used in the teaching of linear algebra.

Students learn to use technological tools.
There is a deluge of software available to help students (and faculty) perform basic (and tedious) computations and visualize advanced concepts. Examples include Matlab, Octave, Mathematica, Maxima, Maple, Sage, Desmos, and GeoGebra.

Because there are already so many options to choose from, and because there is no way of knowing what ground-breaking tools might become available next year, or even next week, this committee will not focus on any one tool, or even on a few. Each department and instructor will have to choose which computation tool to use in their course. We address here only some principles upon which those decisions should be based.

Any computational tool should allow instructors to model and demonstrate a wide variety of applications quickly. However it is all too easy to display sophisticated, interactive pictures while overlooking, even actively hiding, the crucial role of the underlying computations.

Technology should be used to highlight key ideas visually by tapping into the student's native intuition but should not render the underlying computations invisible. For example, the projection of one vector onto another in $\boldsymbol{R}^2$ or $\boldsymbol{R}^3$ can be visualized easily. In higher dimensions this is much harder, but projection in R^n, and in more abstract spaces, is crucial to many applications of linear algebra. Highlighting the role of the underlying computations and concepts needs to be the goal of the visualization because it is through the computations that the concepts can be extended to higher dimensions and become more abstract.

The computational tool chosen should allow students to actively explore and discover patterns, make conjectures, and determine or verify results. For example, it takes only basic experimentation to discover if the determinant function is additive and/or multiplicative. With technological assistance computations such as the factoring the characteristic or minimal polynomial of a 10 x 10 matrix can become straightforward, allowing the students to investigate more deeply than can be done without such assistance.

Linear algebra in its modern form exists because of the advent of modern electronic computational hardware and software. Using such tools should be an integral part of any course in linear algebra, but the focus of the course should always be the essential concepts of the topic. Tools will come and go. Concepts will remain.

Students develop mathematical independence and experience open-ended inquiry.
In order to meaningfully engage in any kind of open-ended investigation a student must be sufficiently independent to follow the problem past the parameters prescribed in the instructions. Conversely a student who is mathematically independent will be willing, and able, to explore beyond the constraints of a given problem even when not instructed to do so. Thus open-ended inquiry and mathematical independence are two sides of the same coin.

The problems inherent to linear algebra lend themselves well to an open-ended inquiry paradigm. For example, a typical problem at the beginning of a first course might be:

"Solve the system:

$3x + 2y = 0$

$2x + 3y = -5$"

Once solved one possible open-ended extension might be

"Use your solution to the previous problem to investigate the solution(s) of:

$3x + 2y + 2z = 0$

$2x + 3y - 2z = -5$"

Investigation of the second problem could lead naturally to the concepts of null space, column space, and linear independence. The instructor's job is then to formalize these ideas with the definitions. Other extensions might allow a nascent understanding of eigenstructure, projection, least-squares and the SVD. Used skillfully, open-ended inquiry will fan the twin flames of mathematical independence and mathematical confidence. It should be used liberally.

Specific Student Learning Outcomes (SLOs) and Student Assessments (SAs)

Student Learning Outcomes

At the end of a first course in linear algebra students should be able to:

SLO 1: Explain/describe	Explain concepts in their own words. For example, explain the meaning of the span of a set of vectors or what it means for a set of vectors to be linearly independent.
SLO 2: Compute	Set up and perform, by hand, small order matrix computations; set up and perform large order computations with software use. For example, solve linear systems (any size); compute eigenvalues and eigenvectors.
SLO 3: Prove	Prove basic results in core topics using definitions and theorems and construct short logical arguments.
SLO 4: Make connections	Make connections with other core concepts. For example, connect linear independence of a set of vectors and invertibility of a matrix.
SLO 5: Visualize	Use visual/graphical means (this includes hand drawings) to make sense of linear algebra concepts. For example, linear independence of three vectors in R^3.
SLO 6: Apply	Apply linear algebra to a variety of problems and understand its impact in real life. For example, use Markov Chains to predict the long-term behavior of a phenomena by finding a certain eigenvector.
SLO 7: Become fluent with abstraction	Read and understand the relationships between definitions, theorems and the techniques of linear algebra. State and understand the key points of definitions of main concepts in linear algebra.
SLO 8: Use technology effectively	Be comfortable with using software for computational and experimental purposes, or visualization, and make sure to understand how it all works.

Table 1

Student Assessment

All assignments, projects, class activities, quizzes, presentations, tests, and other course requirements, should align with the learning outcomes (see Table 1) to assess students' attainment of these outcomes. These outcomes can be integrated inside the Learning Management Systems (LMS) and tied to appropriate rubrics.

Every instructor will base their assessment vehicles on their own goals and criteria. We offer the table below in the hope that it will be useful as an organizational scheme.

	Tests	Projects	Quizzes	etc.
SLO 1				
SLO 2				
SLO 3				

B= Basic level , I= Intermediate level, A= Advanced level , M=Mastery
The criteria for these levels are at the instructor's discretion.

Table 2

Core Topics

Essential Topics

- The solution of systems of linear equations by Gaussian Elimination.
- Linear independence of vectors.
- The span of a set of vectors.
- Matrix algebra (matrix addition, multiplication, and inversion).
- R^n as a vector space (the algebra of vectors, column space, row space, and null space of a matrix).
- Linear transformations.
- Abstract vector spaces
- Eigenvectors and eigenvalues.
- Basis, dimension, and change of basis; diagonalization of a matrix.
- The dot product and orthogonality.
- Least squares.

Optional Topics

- Abstract Inner Products and orthogonality.
- Gram-Schmidt orthogonalization.
- The Singular Value Decomposition.
- Symmetric matrices and orthogonal diagonalization.

Prerequisite Skills and Knowledge

Although many courses taught in American colleges and universities have one or more semesters of calculus as a prerequisite, only a solid understanding of, and skill with, high school level algebra is necessary for the student beginning to study linear algebra. The course can quickly ramp up to a high level of abstraction so a prerequisite of calculus as a way of ensuring mathematical maturity is not necessarily misplaced. However mathematical maturity can be attained via the study of linear algebra as much as from calculus. Departments should consider offering linear algebra alongside the first calculus course. See [Dorier], [Burazin], and [Andrews-Larson].

There are emerging areas of application for linear algebra - data science for example - in which an earlier introduction to linear algebra than is currently the norm would be very beneficial to students. Requiring calculus as a prerequisite for linear algebra can unnecessarily delay the mathematical progress and development of these students with respect to understanding mathematical concepts and techniques that are central to their discipline. This delay can also impede the development of students' mathematical maturity, which must be nurtured and takes time to develop.

Other reasons to remove the calculus prerequisite from the first course in linear algebra include the following:

1. In this committee's experience, mathematical maturity can be gained from taking an introductory linear algebra course as well as it can be gained from taking a calculus course.
2. The linear algebra course will be useful for some students whose major does not require that they take calculus.
3. Taking linear algebra first might impart the mathematical maturity that would help some students do better in calculus.
4. Beginning students may have a better chance of success in a linear algebra course than in a calculus course, helping to smooth their transition from high school mathematics to university mathematics.

References and Resources

[Andrews-Larson] A. Andrews-Larson, J. Siefken, R. Simha, (2022). Report on a US-Canadian Faculty Survey on Undergraduate Linear Algebra. *Notices of the American Mathematical Society,* 69(5):806-812

[Burazin] A. Burazin, V. Jungić, and M. Lovrić, (2020). Three Countries, Two Continents: Common Challenges and Opportunities in Teaching First-Year Mathematics. *Notices of the American Mathematical Society,* 67(1):64–67.

[Carlson] D. Carlson, C. R. Johnson, D. C. Lay, and A. Porter, (1993). The linear algebra curriculum study group recommendations for the first course in linear algebra, *The College Mathematics Journal,* 24(1):41–46.

[Diefenderfer] C. Diefenderfer, D. Hill, S. Axler, N. Neudauer, and D. Strong, (2015). Linear Algebra, in the 2015 CUPM Curriculum Guide to Majors in the Mathematical Sciences, *Published and Distributed by the MAA.*

[Dorier] J. Dorier, A. Robert, J. Robinet, and M. Rogalski, (2000). On a research program about the teaching and learning of linear algebra in first year of French Science University. *International Journal of Mathematical Education in Sciences and Technology*, 31(1), 27–35.

[Stewart] S. Stewart, S. Axler, R. Beezer, E. Boman, M. Catral, G. Harel, J. McDonald, D. Strong, and M. Wawro, (2022). The Linear Algebra Curriculum Study Group (LACSG 2.0) Recommendations, *Notices of the American Mathematical Society,* 69(5):813-819