Obtaining the QR Decomposition by Pairs of Row and Column Operations

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Let $S = \{v_1, v_2, \ldots, v_m\}$ be a set of linearly independent vectors in R^n and let A be the matrix that has these vectors as its columns. The Gram-Schmidt process can be applied to the column vectors to produce a new matrix whose column vectors are orthonormal and whose column space is the same as A's. The process replaces each column of A by a linear combination of that column and its predecessors. If Q denotes the matrix with orthonormal columns, then $A = QR$, where R is an upper-triangular nonsingular matrix. This is the "QR factorization" or "QR decomposition" of A. In this note, we show how to obtain the QR decomposition by using pairs of row and column operations.

Suppose the n by m matrix $A = [a_{ij}]$ has linearly independent columns $v_j = (a_{1j}, a_{2j}, \ldots, a_{nj})^T$ for $j = 1, 2, \ldots, m$. Then $A = [a_{ij}]$ for $i = 1, 2, \ldots, n$. Since A^TA is symmetric, and its diagonal elements are positive, we can use $n(n-1)/2$ pairs of row and column operations on A^TA to annihilate all of its off-diagonal entries. That is, we obtain $B^TA^TAB = \text{diag}[d_1, d_2, \ldots, d_m]$, where B and B^T are products of respective lower-triangular and upper-triangular matrices. Since $B^TA^TAB = (AB)^T(AB) = \text{diag}[d_1, d_2, \ldots, d_m]$, the columns of AB are orthonormal. Letting C be a square root of this diagonal matrix, we have $(C^{-1}B^TA^T)(ABC^{-1}) = I$. Thus, the matrix $Q = ABC^{-1}$ has orthonormal columns. So $A = QR$, where $R = CB^{-1}$ is an upper-triangular nonsingular matrix.

Example. Find the QR decomposition of $A = [v_1, v_2, v_3]$, where $v_1 = (1, 1, -1, 0)^T$, $v_2 = (0, 2, 0, 1)^T$, and $v_3 = (-1, 0, 0, 1)^T$.

Beginning with $A^TA = \begin{bmatrix} 3 & 2 & -1 \\ 2 & 5 & 1 \\ -1 & 1 & 2 \end{bmatrix}$, we apply the following pairs of matrices to remove the off-diagonal entries:

$$B_1[A^TA]B_1^T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \frac{1}{3} & 0 & 1 \end{bmatrix} \begin{bmatrix} 3 & 2 & -1 \\ 2 & 5 & 1 \\ -1 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 & \frac{1}{3} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 2 & 0 \\ 2 & 5 & \frac{5}{3} \\ 0 & \frac{5}{3} & \frac{5}{3} \end{bmatrix}$$

$$B_2[B_1A^TAB_1^T]B_2^T = \begin{bmatrix} 1 & 0 & 0 \\ -\frac{2}{3} & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3 & 2 & 0 \\ 2 & 5 & \frac{5}{3} \\ 0 & \frac{5}{3} & \frac{5}{3} \end{bmatrix} \begin{bmatrix} 1 & -\frac{2}{3} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 0 & 0 \\ 0 & \frac{11}{3} & \frac{5}{3} \\ 0 & \frac{5}{3} & \frac{5}{3} \end{bmatrix}$$

$$B_3[B_2B_1A^TAB_1^TB_2^T]B_3^T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -\frac{5}{11} & 1 \end{bmatrix} \begin{bmatrix} 3 & 0 & 0 \\ 0 & \frac{11}{3} & \frac{5}{3} \\ 0 & \frac{5}{3} & \frac{5}{3} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -\frac{5}{11} \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 0 & 0 \\ 0 & \frac{11}{3} & 0 \\ 0 & 0 & \frac{10}{11} \end{bmatrix}$$

Letting $B = B_1^T B_2^T B_3^T = \begin{bmatrix} 1 & -2/3 & 7/11 \\ 0 & 1 & -5/11 \\ 0 & 0 & 1 \end{bmatrix}$, we have $B^{-1} = \begin{bmatrix} 1 & 2/3 & -1/3 \\ 0 & 1 & 5/11 \\ 0 & 0 & 1 \end{bmatrix}$.

For a square root of the diagonal matrix $\begin{bmatrix} 3 & 0 & 0 \\ 0 & 11/3 & 0 \\ 0 & 0 & 10/11 \end{bmatrix}$, let $C = \begin{bmatrix} \sqrt{3} & 0 & 0 \\ 0 & \sqrt{33}/3 & 0 \\ 0 & 0 & \sqrt{110}/11 \end{bmatrix}$.

Then $C^{-1} = \begin{bmatrix} \sqrt{3}/3 & 0 & 0 \\ 0 & \sqrt{33}/11 & 0 \\ 0 & 0 & \sqrt{110}/10 \end{bmatrix}$, and we obtain

$$Q = ABC^{-1} = \begin{bmatrix} \frac{\sqrt{3}}{3} & \frac{-2\sqrt{33}}{33} & \frac{-2\sqrt{110}}{55} \\ \frac{\sqrt{3}}{3} & \frac{4\sqrt{33}}{33} & \frac{-3\sqrt{110}}{110} \\ \frac{-\sqrt{3}}{3} & \frac{2\sqrt{33}}{33} & \frac{-7\sqrt{110}}{110} \\ 0 & \frac{\sqrt{33}}{11} & \frac{3\sqrt{110}}{55} \end{bmatrix}$$

and

$$R = CB^{-1} = \begin{bmatrix} \sqrt{3} & \frac{2\sqrt{3}}{3} & \frac{-\sqrt{3}}{3} \\ 0 & \frac{\sqrt{33}}{3} & \frac{5\sqrt{33}}{33} \\ 0 & 0 & \frac{\sqrt{110}}{11} \end{bmatrix}.$$

Although the foregoing method of orthonormalizing a matrix is not frequently used, as compared to the standard Gram-Schmidt process, its procedure is simple, and it is able to avoid the inaccuracy problems inherent in the latter method.

This note can be viewed as an explication of the idea found in section 4 of [**1**], in which many references contain detailed information on the decomposition of real and complex matrices.

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Reference

1. Roger A. Horn and Ingram Olkin, When does $A * A = B * B$ and why does one want to know, *American Mathematical Monthly* **103** (1996) 470–482.

The n^{th} Derivative Test and Taylor Polynomials Crossing Graphs

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In [**2**], Samuel B. Johnson develops a nice criterion for determining when the graphs of Taylor polynomials and their associated functions will cross.