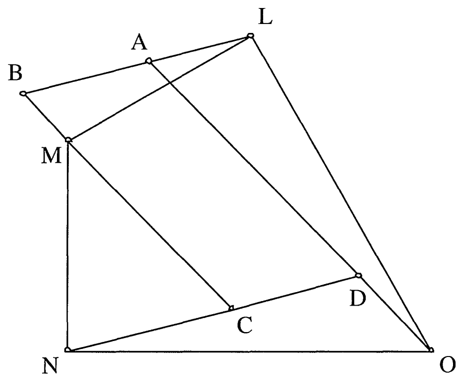

Figure 6 The kite problem

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Cramer's Rule Is Due To Cramer

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Most freshmen who took the first calculus course know of Marquis de l'Hôpital as the man who did not invent l'Hôpital's rule. There is a certain core of truth to this assertion, but it is nevertheless somewhat unfair to the man, who was a productive mathematician in his time, highly respected by the Bernoullis and Leibniz, and an author of two excellent textbooks on calculus and analytic geometry.

It seems that the same fate now threatens Gabriel Cramer, who is in danger of becoming the man who did not invent Cramer's rule. Some authors[1] credit this invention to Colin Maclaurin on the basis of his *Treatise of Algebra*, edited from various manuscripts left at his death and published posthumously in 1748. (Recently B. Hedman [**4**] found among the unpublished papers of Maclaurin a manuscript of Part I of the *Treatise* dated 1729 and almost identical to the 1748 edition.) The *Treatise* was intended both as a textbook and as a sort of supplement to Newton's *Arithmetic*, pro-

[1]This claim has its origin in a note by C. B. Boyer [**1**]. M. Kline [**5**] asserts that "the solution of linear equations in two, three, and four unknowns by the method of determinants was created by Maclaurin."

viding proofs to various assertions that Newton did not bother to prove.[2] That it did, and much else besides. But one thing that it did not do was to provide Cramer's rule.

Let us review the evidence.

What is known as Cramer's rule is a formula expressing solutions of a system of n linear equations with n unknowns as a ratio of two quantities, each of which is a sum of products of certain coefficients provided with appropriate signs. The rule for forming the products is not difficult to state, especially for the denominator of the ratio, which is the same for all unknowns. The rule for signs, the heart of the matter, is almost impossible to state without an appropriate indexing of unknowns. Such indexing was already introduced by Leibniz in a 1693 letter to l'Hôpital and in 1700 in *Acta Eruditorum*, but it seems that nobody noticed.

Maclaurin certainly did not notice. In his *Algebra* (pp. 82–85) he solves, first, a system of two equations with two unknowns, then a system of three equations. In both cases coefficients are given by unindexed small letters $a, b, c, \ldots$. Both solutions are arrived at by elimination. The solution of three equations is followed by a discussion of a rule for forming denominators and numerators. With a proper interpretation of his slightly confusing definition of "opposite" coefficients, the rule becomes correct—but for the lack of the convention concerning signs. This is followed by a breezy assurance that systems of four equations can be solved "much after the same manner by taking all the products that can be made of four opposite coefficients and always prefixing contrary signs to those that involve the products of two opposite coefficients." Since Maclaurin calls two coefficients opposite if they are attached to distinct unknowns in distinct equations, *every* product involves "the products of two opposite coefficients," and this "rule" makes no sense whatever. The most charitable explanation is that it is an attempt to describe what happens in the case of a system of two equations when two products are provided with "contrary signs." However, without stating which signs have to be affixed to which monomials, the "rule" is not adequate even in this case. In the general case, it indicates that Maclaurin did not know the correct rule for signs.

No more is said about linear equations in the *Treatise*; in particular, no notice is taken of the possibility that the denominator may vanish, rendering the formulas meaningless.

It would be incorrect to attach much blame to Maclaurin for this muddle. In the middle of the 18th century the solution by elimination of systems of linear equations did not present a problem to which a mathematician of Maclaurin's class would attach much attention. He was writing a textbook, and, in a hurry to get to some really interesting stuff, he certainly missed an opportunity to discover Cramer's rule.[3]

That much about Maclaurin. We now go to Gabriel Cramer, a Swiss mathematician known for his excellent editions of the works of James Bernoulli (2 vols., Geneva, 1744), and of John Bernoulli (4 vols., Geneva, 1742). However, his best known work is a hefty volume of 680 pages in-quarto, entitled *Introduction à l'Analyse des Lignes Courbes Algébriques* and published in Geneva in 1750. It is a well-organized and well-written book that contains most of what was known at the time about algebraic geometry, as well as Cramer's original contributions to the subject. In the appendix to this work, pp. 657–659 are devoted to a concise exposition of the theory of systems of

[2]Providing commentaries to the *Arithmetic* appears to have been a popular occupation in the eighteenth century. The first edition of the *Arithmetic* had a supplement by Halley. It was removed by Newton from the second (Latin) edition prepared by him in 1722, but reappeared, together with 7 other commentaries, in the edition published by s'Gravesande in 1732. The next edition commented by Castillione grew to two volumes in-quarto and the number of commentaries to 9.

[3]Already in 1901, M. Cantor [**2**] noted that, for lack of good notation, Maclaurin missed the general rule for solving linear equations. He also reproduced Cramer's solution [**2**, p. 607].

linear equations. The presentation is very clear. It may be summarized, with inessential modifications and retaining Cramer's notation, as follows.

We consider the system of n equations for n unknowns z, x, y, v, &c.:

$$A_1 = Z_1 z + Y_1 y + X_1 x + V_1 v + \&c.$$
$$A_2 = Z_2 z + Y_2 y + X_2 x + V_2 v + \&c.$$
$$A_3 = Z_3 z + Y_3 y + X_3 x + V_3 v + \&c.$$
$$\&c.$$

It is agreed that the letter Z always denotes the coefficient of the first unknown, the letter Y that of the second unknown, and so on.[4]

To find the solution, first form all expressions that can be obtained from the product $Z \quad Y \quad X \quad V \ldots$ (always in this order) by distributing as lower indices all permutations of numbers $1, 2, \ldots, n$. (For example, with $n = 2$ one obtains two terms: $Z_1 Y_2$ and $Z_2 Y_1$.) Now, count the number of transpositions (*dérangements*) in the permutation attached to a given term. If it is odd, then the term is provided with the minus sign, otherwise with the plus sign. The solution of the system is given by fractions which have as the denominator the sum of terms just obtained, and as a numerator the sum of terms formed, for the unknown z, by replacing the letter Z by A, for the unknown y, the letter Y by A, and so on.

This, of course, is Cramer's rule. Cramer did not provide a proof. However, he did consider what happens if the denominator vanishes (that is, in today's terminology, if the rank of the matrix of coefficients is less than n). He split this case into two according to whether the rank of the augmented matrix equals n or is less than n, and showed that in the first case the system will have no solution. The second case was called "indeterminate" and left at that.

Thus, Cramer's rule is due to Cramer. In fact, more is due to him. The procedure given above for attaching a number to a square array of numbers, of arbitrary size, is effectively the first definition of a determinant. Of course, any formula for a solution of a system of 2 equations with two unknowns and literal coefficients, whether arrived at by elimination or by any other method, will express this solution as a ratio of two determinants. Also, it is known that determinants of 2×2, 3×3, and 4×4 arrays were calculated earlier and in a different context by the Japanese mathematician Seki Takakazu, (see a note by Victor Katz in Fraleigh and Beauregard's *Linear Algebra* [**3**, p. 251]). But the first unambiguous, general definition of determinants of arbitrary size is due to Cramer. A formal recognition of this fact and the name of the object defined is missing in his book. This was provided later and is another story.

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[4]This system of indexing is different from Leibniz's. However, as the editor of works of both Bernoullis, Cramer was certainly well acquainted with everything that had been published in *Acta Eruditorum*.