Proof Without Words: How Did Archimedes Sum Squares in the Sand?

KATHERINE KANIM
New Mexico State University
Las Cruces, NM 88003

Archimedes deduced a formula for a sum of squares, used in his determination of the volume of a conoid (*Conoids and Spheroids*, Prop. 25) and areas of spirals (*Spirals*, Prop. 10 and Prop. 24). The modern transcriptions of his proof (Dijksterhuis, Heath, Heiberg) are completely algebraic and hard to follow. The geometry of Archimedes' proof is depicted visually below. Each step of his written proof is transparent in the geometry of the picture. One wonders if this is the picture Archimedes drew in the sand.

Spirals, Proposition 10 (Dijksterhuis, page 122)

> If a series of any number of lines be given, which exceed one another by an equal amount, and the difference be equal to the least, and if other lines be given equal in number to these and in quantity to the greatest, the squares on the lines equal to the greatest, plus the square on the greatest and the rectangle contained by the least and the sum of all those exceeding one another by an equal amount will be the triplicate of all the squares on the lines exceeding one another by an equal amount.

$$(n+1)n^2 + \sum_{i=1}^{n} i = 3\sum_{i=1}^{n} i^2$$

Archimedes also deduced corollary inequalities, which he used for his area and volume proofs by the method of exhaustion. Can you see them in the pictures?

$$n \cdot n^2 < 3\sum_{i=1}^{n} i^2$$

$$n \cdot n^2 > 3\sum_{i=1}^{n-1} i^2$$

Acknowledgment. Thank you to David Pengelley for suggestions and support, to Steve Kanim for the electronic version of the figure, and to the referee for ideas about the layout of the figure.

REFERENCES

1. E. J. Dijksterhuis, *Archimedes*, Princeton University Press, Princeton, NJ, 1987.
2. Thomas L. Heath, *The Works of Archimedes*, in Great Books of the Western World, vol. 11, Encyclopaedia Britannica, Inc., Chicago, IL, 1952.
3. J. L. Heiberg, *Archimedes Opera*, vol. 2, Teubner, Leipzig, 1880–81.