Two Irrational Numbers From the Last Nonzero Digits of $n!$ and n^n

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We begin by looking at the pattern formed from the last (that is, units) digit in the base 10 expansion of n^n. Since $1^1 = 1$, $2^2 = 4$, $3^3 = 27$, $4^4 = 256$, and so on, we can easily calculate the first few numbers in our pattern to be $1, 4, 7, 6, 5, 6, 3, 6 \ldots$. We construct a decimal number $N = 0.d_1d_2d_3 \ldots d_n \ldots$ such that the n^{th} digit d_n of N is the last (i.e. unit) digit of n^n; that is, $N = 0.14765636\ldots$. In a recent paper [**1**], R. Euler and J. Sadek showed that this N is a rational number with a period of twenty digits:

$$N = 0.\overline{14765636901636567490}.$$

This is a nice result, and we might well wonder if it can be extended. Indeed, Euler and Sadek [**1**] recommend looking at the last *nonzero* digit of $n!$ (If we just looked at the last digit of $n!$, we would get a very dull pattern of all 0s, as $n!$ ends in 0 for every $n \geq 5$.)

With this is mind, let's define $\text{lnzd}(A)$ to be the last nonzero digit of the positive integer A; it is easy to see that $\text{lnzd}(A) \equiv A/10^i \bmod 10$, where 10^i is the largest power of 10 that divides A. We wish to investigate not only the pattern formed by $\text{lnzd}(n!)$, but also the pattern formed by $\text{lnzd}(n^n)$. In accordance with Euler and Sadek [**1**], we define the *factorial number*, $F = 0.d_1d_2d_3 \ldots d_n \ldots$ to be the infinite decimal such that each digit $d_n = \text{lnzd}(n!)$; similarly, we define the *power number*, $P = 0.d_1d_2d_3 \ldots d_n \ldots$ by $d_n = \text{lnzd}(n^n)$. We ask whether these numbers are rational or irrational.

Although the title of this article gives away the secret, we'd like to point out that at first glance, our factorial number F exhibits a suprisingly high degree of regularity, and a fascinating pattern occurs. The first few digits of F are easy to calculate:

$1! = \underline{1}$	$5! = 1\underline{2}0$	$10! = 3628\underline{8}00$
$2! = \underline{2}$	$6! = 7\underline{2}0$	$11! = 39916\underline{8}00$
$3! = \underline{6}$	$7! = 50\underline{4}0$	$12! = 479001\underline{6}00$
$4! = 2\underline{4}$	$8! = 403\underline{2}0$	$13! = 6227020\underline{8}00$
	$9! = 3628\underline{8}0\ldots$	$14! = 87178291\underline{2}00$

Reading the underlined digits, we have

$$F = 0.1264\ 22428\ 88682\ldots.$$

Continuing along this path, we have (to forty-nine decimal places)

$$F = 0.1264\ 22428\ 88682\ 88682\ 44846\ 44846\ 88682\ 22428\ 22428\ 66264\ldots.$$

It is not hard to show that (after the first four digits) F breaks up into five-digit blocks of the form $x\ x\ 2x\ x\ 4x$, where $x \in \{2, 4, 6, 8\}$, and the $2x$ and $4x$ are taken mod 10. Furthermore, if we represent these five-digit blocks by symbols ($\dot{2}$ for 22428, $\dot{4}$ for

44846, $\dot{6}$ for 66264, $\dot{8}$ for 88682, and $\dot{1}$ for the initial four-digit block of 1264), we have

$$F = 0.\dot{1} \quad \dot{2} \quad \dot{8} \quad \dot{8} \quad \dot{4} \quad \dot{4} \quad \dot{8} \quad \dot{2} \quad \dot{2} \quad \dot{6} \quad \ldots.$$

Grouping these symbols into blocks of five and then performing more calculations (with the aid of *Maple*) give us F to 249 decimal places:

$$F = 0.\dot{1}\dot{2}\dot{8}\dot{8}\dot{4}\ \dot{4}\dot{8}\dot{2}\dot{2}\dot{6}\ \dot{2}\dot{4}\dot{6}\dot{6}\dot{8}\ \dot{4}\dot{8}\dot{2}\dot{2}\dot{6}\ \dot{4}\dot{8}\dot{2}\dot{2}\dot{6}\ \dot{8}\dot{6}\dot{4}\dot{4}\dot{2}\ \dot{2}\dot{4}\dot{6}\dot{6}\dot{8}\ \dot{6}\dot{2}\dot{8}\dot{8}\dot{4}\ \dot{2}\dot{4}\dot{6}\dot{6}\dot{8}\ \dot{2}\dot{4}\dot{6}\dot{6}\dot{8}\ \ldots.$$

The reader will notice additional patterns in these blocks of five symbols (twenty-five digits). In fact, such patterns exist for any block of size 5^i. However, a pattern is different from a period, and doesn't imply that our decimal F is rational. Consider the classic example of 0.1 01 001 0001 00001 000001 ..., which has an obvious pattern but is obviously irrational. It turns out that our decimal F is also irrational, as the following theorem indicates:

THEOREM 1. *Let* $F = 0.d_1d_2d_3\ldots d_n\ldots$ *be the infinite decimal such that each digit* $d_n = \mathrm{lnzd}(n!)$. *Then, F is irrational.*

We will prove this, but first note that our power number, P, might also seem to be rational at first glance. P is only slightly different from Euler and Sadek's rational number N, as seen here:

$$N = 0.14765\ 6369\underline{0}\ 16365\ 6749\underline{0}\ 14765\ 6369\underline{0}\ 16365\ 6749\underline{0}\ldots$$

$$\text{and}\quad P = 0.14765\ 6369\underline{1}\ 16365\ 6749\underline{6}\ 14765\ 6369\underline{9}\ 16365\ 6749\underline{6}\ldots$$

(Again, calculations were performed by *Maple*.) Despite this striking similarity between P and N, it turns out that P, like F, is irrational:

THEOREM 2. *Let* $P = 0.d_1d_2d_3\ldots d_n\ldots$ *be the infinite decimal such that each digit* $d_n = \mathrm{lnzd}(n^n)$. *Then, P is irrational.*

Before we begin with the (slightly technical) proofs, let us pause to get a feel for why these two numbers must be irrational. There is no doubt that both F and P are highly regular, in that both exhibit a lot of repetition. The problem is that there are *too many* patterns in the digits, acting on different scales. Taking P, for example, we note that there is an obvious pattern (as shown by Euler and Sadek in [**1**]) repeating every 20 digits with $1^1, 2^2, 3^3, \ldots, 9^9$ and $11^{11}, 12^{12}, \ldots, 19^{19}$, but this is broken by a similar pattern for $10^{10}, 20^{20}, \ldots, 90^{90}$ and $110^{110}\ldots 190^{190}$, which repeats every 200 digits. This, in turn, is broken by another pattern repeating every 2000, and so on. A similar behaviour is found for F, but in blocks of 5, 25, 125, and so on, as mentioned above. So, in vague terms, there are always new patterns starting up in the digits of P and of F, and this is what makes them irrational.

Are there some simple observations that we can make about P and F to help us to prove our theorems? To start with, we might notice that every digit of F (except for the first one) is even. Can we prove this? Yes, and without much difficulty:

LEMMA 1. *For* $n \geq 2$, *then* $\mathrm{lnzd}(n!)$ *is in* $\{2, 4, 6, 8\}$.

Proof. The lemma is certainly true for $n = 2, 3, 4$. For $n \geq 5$, we note that the prime factorization of $n!$ contains more 2s than 5s, and thus even after taking out all the 10s in $n!$, the quotient will still be even. To be precise, the number of 5s in $n!$ (and thus the number of trailing zeros in its base-10 representation) is $e_5 = \sum_{i=1}^{\infty} \left[n/5^i\right]$, which is strictly less than the number of 2s, $e_2 = \sum_{i=1}^{\infty} \left[n/2^i\right]$ (here, $[\cdot]$ represents the greatest integer function). Hence, $n!/10^{e_5}$ is an even integer not divisible by 10, and so $\text{lnzd}(n!) \equiv n!/10^{e_5} \bmod 10$, which must be in $\{2, 4, 6, 8\}$. ■

Another helpful observation is that the lnzd function is at least sometimes multiplicative. For example,

$$\begin{aligned} &\text{lnzd}(12) \cdot \text{lnzd}(53) = 2 \cdot 3 = 6, \\ \text{and}\quad &\text{lnzd}(12 \cdot 53) = \text{lnzd}(636) = 6. \end{aligned}$$

However, we note that at times this would-be rule fails:

$$\begin{aligned} &\text{lnzd}(15) \cdot \text{lnzd}(22) = 5 \cdot 2 = 10, \\ \text{yet}\quad &\text{lnzd}(15 \cdot 22) = \text{lnzd}(330) = 3. \end{aligned}$$

So, we can only prove a limited form of multiplicativity, but it is useful none the less:

LEMMA 2. *Suppose a, b are integers with* $\text{lnzd}(a) \neq 5$, $\text{lnzd}(b) \neq 5$. *Then,* lnzd *is multiplicative; that is,* $\text{lnzd}(a \cdot b) \equiv \text{lnzd}(a) \cdot \text{lnzd}(b)$ *mod* 10.

Proof. Let x' denote the integer x without its trailing zeros; that is, $x' = x/10^i$, where 10^i is the largest power of 10 dividing x. (Note that $\text{lnzd}(x) \equiv x' \bmod 10$.) By hypothesis, a' and b' are both $\neq 0 \bmod 5$, and so $(a \cdot b)' \neq 0 \bmod 5$ and so $(a \cdot b)' = a' \cdot b'$. Thus,

$$\begin{aligned} \text{lnzd}(a \cdot b) &= \text{lnzd}((a \cdot b)') = \text{lnzd}(a' \cdot b') \equiv a' \cdot b' \bmod 10, \quad \text{while} \\ \text{lnzd}(a) \cdot \text{lnzd}(b) &= \text{lnzd}(a') \cdot \text{lnzd}(b') = (a' \bmod 10) \cdot (b' \bmod 10). \end{aligned}$$

The two are clearly congruent mod 10. ■

We are now ready to prove Theorem 1, to show that F is irrational. The proof is a little technical; it proceeds by assuming that F has a repeating decimal expansion with period λ_0, then choosing an appropriate multiple of λ_0 and an appropriate digit d, in order to arrive at a contradiction.

Proof of Theorem 1: Suppose F is rational, and thus eventually periodic. Let λ_0 be the period, so that for every n sufficiently large, then $d_n = d_{n+\lambda_0}$. Write $\lambda_0 = 5^i \cdot K$ such that $5 \nmid K$ (we acknowledge that K could be 1) and let $\lambda = 2^i \cdot \lambda_0 = 10^i \cdot K$. Then, $\text{lnzd}(\lambda) = \text{lnzd}(K)$, and since $5 \nmid K$, then $10 \nmid K$ and so $\text{lnzd}(K) \equiv K \bmod 10$. Note also that $\text{lnzd}(2\lambda) \equiv 2K \bmod 10$. Choose M sufficiently large so that both of the following are true: $\text{lnzd}(10^M + \lambda) = \text{lnzd}(\lambda)$ (this can easily be done by demanding that $10^M > \lambda$), and $d_n = d_{n+\lambda_0}$ for all $n \geq M$. Finally, let $d = \text{lnzd}((10^M - 1)!)$; since $10^M! = (10^M - 1)! \cdot 10^M$, then d also equals $\text{lnzd}(10^M!)$.

Since λ is a multiple of the period λ_0, if we let $A = 10^M - 1 + \lambda$ and $B = 10^M - 1 + 2\lambda$, then

$$\begin{aligned} &d = \text{lnzd}((10^M - 1)!) = \text{lnzd}(A!) = \text{lnzd}(B!) \\ \text{and}\quad &d = \text{lnzd}(10^M!) = \text{lnzd}((A+1)!) = \text{lnzd}((B+1)!). \end{aligned}$$

We will find our contradiction in the last two terms in the above equation. By Lemma 1, $d \in \{2, 4, 6, 8\}$, and so $\text{lnzd}(A!) \neq 5$. Also, since $\text{lnzd}(A+1) = \text{lnzd}(10^M + \lambda) = \text{lnzd}(\lambda) \equiv K \bmod 10$, we know that $\text{lnzd}(A+1) \neq 5$. Thus, we can apply Lemma 2 to $\text{lnzd}(A! \cdot (A+1))$ to get

$$d = \text{lnzd}((A+1)!) = \text{lnzd}(A!) \cdot \text{lnzd}(A+1) \equiv d \cdot K \bmod 10.$$

Likewise, working with B, we find

$$d = \text{lnzd}((B+1)!) = \text{lnzd}(B!) \cdot \text{lnzd}(B+1) \equiv d \cdot 2K \bmod 10.$$

Combining these two equations, we get $d(1-K) \equiv d(1-2K) \equiv 0 \bmod 10$. Since $5 \nmid d$, this implies that $5 | (1-K)$ and $5 | (1-2K)$, which is a contradiction. Thus, there can be no period λ_0 and so F is irrational. ■

We now turn our attention to the power number P derived from the last nonzero digits of n^n. This part was more difficult, but a major step was the discovery that the sequence $\text{lnzd}(100^{100})$, $\text{lnzd}(200^{200})$, $\text{lnzd}(300^{300}) \ldots$ was the same as the sequence $\text{lnzd}(100^4)$, $\text{lnzd}(200^4)$, $\text{lnzd}(300^4) \ldots$. This relies not only on the fact that $4|100$ but also on the easily proved fact that $a^b \equiv a^{b+4} \bmod 10$ for $b > 0$, used in the following lemma:

LEMMA 3. *Suppose* $100 \mid x$. *Then,* $\text{lnzd}(x^x) \equiv (\text{lnzd}\, x)^4 \bmod 10$.

Proof. As in Lemma 2, let x' denote the integer x without its trailing zeros; that is, $x' = x/10^i$, where 10^i is the largest power of 10 dividing x. Now,

$$\begin{aligned}\text{lnzd}(x^x) &= \text{lnzd}((10^i x')^{10^i x'}) \\ &= \text{lnzd}((10^{i \cdot 10^i x'})(x')^{10^i \cdot x'}) \\ &= \text{lnzd}((x')^{10^i \cdot x'}).\end{aligned}$$

Since $10 \nmid x'$, then $10 \nmid (x')^{10^i \cdot x'}$, and so

$$\text{lnzd}(x^x) \equiv (x')^{10^i \cdot x'} \bmod 10.$$

Since $100 \mid x$, then $4 \mid 10^i \cdot x'$, and since $(x')^n \equiv (x')^{n+4} \bmod 10$ for every positive n, we can repeatedly reduce the exponent of x' by 4 until we have

$$\begin{aligned}\text{lnzd}(x^x) &\equiv (x')^4 \bmod 10 \\ &\equiv (\text{lnzd}\, x)^4 \bmod 10.\end{aligned}$$ ■

With Lemma 3 at our disposal, the proof of Theorem 2 is now fairly easy.

Proof of Theorem 2: Again, we argue by contradiction. Suppose P is rational. Let λ_0 be the eventual period, and choose j sufficiently large such that $10^j > 200 \cdot \lambda_0$ and such that

$$\text{lnzd}((10^j + n\lambda_0)^{10^j + n\lambda_0}) = \text{lnzd}((10^j)^{10^j})$$

for every positive n. Choosing $n = 200$, we get

$$\text{lnzd}((10^j + 200\lambda_0)^{10^j + 200\lambda_0}) = \text{lnzd}((10^j)^{10^j}).$$

We reduce the left side of the above equation by Lemma 3 (note that $\text{lnzd}(10^j + 200\lambda_0) = \text{lnzd}(2\lambda_0)$), and the right side is obviously 1, so we have

$$(\text{lnzd}\, 2\lambda_0)^4 \equiv 1 \bmod 10$$

Note that $\text{lnzd}(2\lambda_0)$ can only be 2, 4, 6, or 8, and raising these to the fourth power mod 10 gives us the contradiction $6 = 1$. Thus, P is irrational. ■

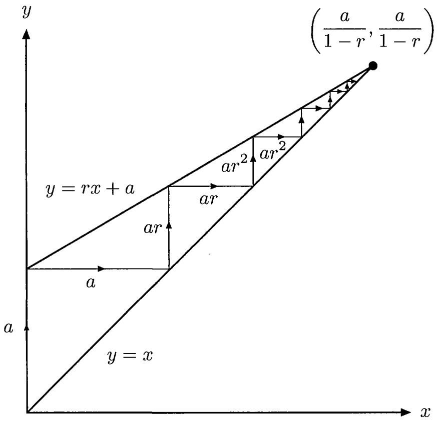
The obvious next question is far more difficult: Are F and P algebraic or transcendental? I suspect the latter, but it is only a hunch. Perhaps some curious reader will continue along this interesting line of study.

REFERENCES

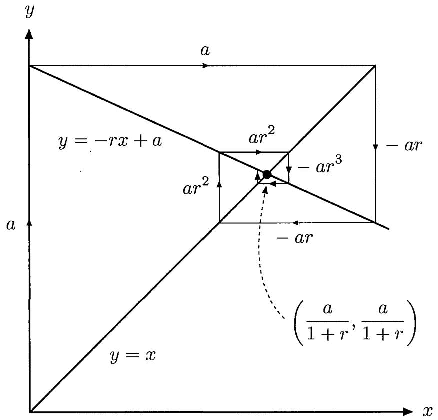
1. R. Euler and J. Sadek, A number that gives the unit digit of n^n, *Journal of Recreational Mathematics*, **29** (1998) No. 3, pp. 203–4.

Proof without Words: Geometric Series

THE VIEWPOINTS 2000 GROUP*



$a + ar + ar^2 + \cdots = \frac{a}{1-r}, \quad 0 < r < 1.$



$a - ar + ar^2 - \cdots = \frac{a}{1+r}, \quad 0 < r < 1.$

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