CLASSROOM CAPSULES

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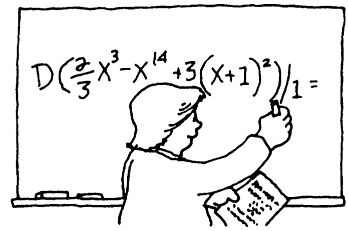

A Classroom Capsule is a short article that contains a new insight on a topic taught in the earlier years of undergraduate mathematics. Please submit manuscripts prepared according to the guidelines on the inside front cover to Tom Farmer.

Additivity ⊕ Homogeneity

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As we all know, a linear transformation is a function T on a vector space V that has the two properties,

additivity: $T(u+v) = T(u) + T(v)$ for all $u, v \in V$, and

homogeneity: $T(\alpha v) = \alpha T(v)$ for all $v \in V$ and all scalars α.

What functions have one of the properties but not the other? The question arose when one of our students conjectured that any homogeneous function would necessarily be additive. Since linear transformation is one of the central ideas in linear algebra, one would expect that most textbooks on the subject would contain examples, but we were not able to find a single one in any of the thirty texts we examined! So, we suggest the question as a discussion or research problem for students in linear algebra or in introduction-to-proof courses. Asking students to discover some examples and describe general classes of them can be an inviting way to familiarize students with the workings of linear transformations and to engage them in creative mathematical research.

As a starting point, there is

$$T_1(x, y) = \sqrt[3]{x^3 + y^3}\,.$$

It is homogeneous because $T_1(\alpha(x, y)) = \sqrt[3]{(\alpha x)^3 + (\alpha y)^3} = \alpha\sqrt[3]{x^3 + y^3} = \alpha T_1(x, y)$, but it is not additive because $T_1(1,0) + T_1(0,1) = 1 + 1 = 2$ while

$T_1(1,1)=\sqrt[3]{2}$. This example suggests taking the nth root of a homogeneous polynomial of odd degree n, such as

$$T_2(x,y,z)=\sqrt[5]{x^5-x^2y^3+xy^2z^2}\,.$$

The quantity inside the nth root could be anything that is homogeneous of order n, as

$$T_3(A)=\sqrt[7]{|A|}$$

where A is a 7×7 matrix, or

$$T_4\big(a_0+a_1x+a_2x^2+\cdots+a_nx^n\big)=\sqrt[9]{a_0a_1\ldots a_8}\,.$$

A second way to generalize the example is to write it as

$$T_1(x,y)=\begin{cases} x\sqrt[3]{1+y^3/x^3} & \text{if } x\neq 0\\ y & \text{if } x=0.\end{cases}$$

In fact, if f is not linear, then

$$T_5(x,y)=\begin{cases} xf(y/x) & \text{if } x\neq 0\\ cy & \text{if } x=0\end{cases}$$

is homogeneous but not additive. This idea can be readily generalized to higher dimensions. Other examples of such functions can be built from pairs of homogeneous functions, as

$$T_6(x,y)=\begin{cases} \operatorname{tr}(A) & \text{if } A \text{ is singular}\\ \sqrt[3]{|A|} & \text{if } A \text{ is nonsingular}\end{cases}$$

where A is a 3×3 matrix.

Functions that are additive but not homogeneous are harder to find. On real vector spaces, additive functions are necessarily homogeneous for rational scalars (a common textbook exercise) and continuous additive functions are always homogenous (a more advanced result). Thus, an additive non-homogeneous map between real vector spaces would have to be homogeneous for rational scalars without being continuous. For a construction of such a function using Zorn's Lemma, see [**1**, p. 20]. However, complex vector spaces readily provide examples, as

$$T_7(z)=\operatorname{Re}(z)\ \text{or}\ T_8(z)=\bar{z}.$$

Both functions are clearly additive, but they fail to be homogeneous for complex scalars. They suggest the class of functions

$$T_9(z)=c_1\operatorname{Re}(z)+c_2 i\operatorname{Im}(z)$$

for complex numbers $c_1\neq c_2$.

We have not nearly described all the classes of functions that answer our original question; many other classes exist. Students who discover, generalize, and classify such vector space functions are likely to develop a deeper understanding of linear transformations, and will gain an appreciation of the open-ended nature of mathematical research.

Reference

1. A. Torchinsky, *Real Variables*, Addison-Wesley, 1988.

On "Rethinking Rigor in Calculus...," or Why We Don't Do Calculus on the Rational Numbers

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In a recent "Point/Counterpoint" in the *American Mathematical Monthly* (**[1]**, **[2]**), it was suggested that the basic theorems on continuous functions and their derivatives (the Boundedness Theorem, the Extreme Value Theorem, the Intermediate Value Theorem, and, especially, the Mean Value Theorem) be omitted from the introductory calculus course. Reasons given were that "the origin of the Mean Value Theorem in the structure of the real numbers ... is too difficult for a standard course"; that these discussions are "the sort of thing that gives mathematics a bad name: assuming the nonobvious to prove the obvious"; that perhaps there is no "need for formal theorems and proofs in a standard calculus course"; and that, in any event, one shouldn't "prove things in more generality than is necessary; even analysts don't usually deal with the discontinuous derivatives allowed by the Mean Value Theorem."

I demur. Without commenting on the pedagogical issues, I would like to point out that this program risks serious misdirection of the mathematical intuition of its students. In particular, I submit that the notion that these basic theorems are "obvious," save for obscure subtleties raised only by bizarre, pathological functions (which are scarcely encountered in practice) is incorrect.

A quick glance at the standard proofs of these basic theorems on continuous functions shows that they represent direct (or nearly direct) applications of the Axiom of Completeness as applied to their *domain*—that is, they reflect the existence of particular limit points guaranteed by the Axiom of Completeness, acting on the domain of a continuous, real-valued function. One way to see what is going on is to consider continuous functions on an *incomplete* domain, say the set of rational numbers, $\mathbf{Q}$.

Of course, it is important to remember that continuity depends only on the points where a function is defined — that is to say, on the points *in the domain* of the function. Many of the examples that follow have been chosen to highlight the "hole" in the rational number line at $1/\sqrt{2}$, in recognition of the historic role of $\sqrt{2}$ as perhaps the first number shown to be irrational. Note that it is possible to