The Matrix of a Rotation

Roger C. Alperin, San Jose State University, San Jose, CA

What is the matrix of the rotation of R^3 about a unit axis $\mathbf{p}$ through an angle θ? Since a rotation has a fixed axis (or eigenvector of eigenvalue 1) and rotates the plane perpendicular to $\mathbf{p}$ by angle θ, the matrix is easy to determine if we change to a convenient basis; however, it is not well known that there is a simple expression for the matrix in the standard basis which depends only on the coordinates of $\mathbf{p}$ and the angle θ. The formula is obtained without changing bases. Furthermore, this formula can be useful in coding the effect of a rotation in computer graphics. The derivation of this formula was motivated by the close relationship between rotations and quaternions.

For two vectors $\mathbf{v}$ and $\mathbf{w}$ we use the notation $\mathbf{v} \cdot \mathbf{w}$ for the standard inner product and $\mathbf{v} \times \mathbf{w}$ for the cross product. Consider the linear transformation of R^3 given by $P(\mathbf{q}) = \mathbf{p} \times \mathbf{q}$ where $\mathbf{p}$ is a unit vector.

Proposition 1. $P^2(\mathbf{q}) = -\mathbf{q} + (\mathbf{p} \cdot \mathbf{q})\mathbf{p}$. *Thus* $I + P^2$ *is the projection operator along the unit vector* $\mathbf{p}$.

Proof. The first part follows easily by using the triple product formula $\mathbf{p} \times (\mathbf{q} \times \mathbf{r}) = (\mathbf{p} \cdot \mathbf{r})\mathbf{q} - (\mathbf{p} \cdot \mathbf{q})\mathbf{r}$. The second statement follows easily from the first part and the definition of an orthogonal projection.

Proposition 2. *The rotation of* R^3 *about an axis* $\mathbf{p}$ *of unit length by an angle* θ *is given by*

$$L(\mathbf{q}) = \mathbf{q} + (\sin\theta)P(\mathbf{q}) + (1 - \cos\theta)P^2(\mathbf{q}).$$

Proof. To see this, we show that this linear transformation has the geometric properties of a rotation as described in the first paragraph. The vector $\mathbf{p}$ is left fixed since $P(\mathbf{p}) = P^2(\mathbf{p}) = 0$. It follows from the definition of L and Proposition 1 that if $\mathbf{q}$ is perpendicular to $\mathbf{p}$ then $L(\mathbf{q}) = \cos\theta\mathbf{q} + \sin\theta(\mathbf{p} \times \mathbf{q})$. Moreover if $\mathbf{q}$ is also of unit length then $\mathbf{p} \times \mathbf{q}$ is of unit length and also perpendicular to $\mathbf{p}$; hence

$$L(\mathbf{p} \times \mathbf{q}) = \cos\theta(\mathbf{p} \times \mathbf{q}) + \sin\theta(\mathbf{p} \times (\mathbf{p} \times \mathbf{q})) = \cos\theta(\mathbf{p} \times \mathbf{q}) - \sin\theta\mathbf{q}.$$

Thus the plane perpendicular to $\mathbf{p}$ is rotated by angle θ. It follows now that all of R^3 is rotated about the axis $\mathbf{p}$ by angle θ and thus L describes the rotation.

We can now easily write the matrix of $L = I + (\sin\theta)P + (1 - \cos\theta)P^2$ in terms of the standard basis $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$. Suppose $\mathbf{p} = (a, b, c)^t$; then P is easily computed: $P(\mathbf{e}_1) = (0, c, -b)^t$, $P(\mathbf{e}_2) = (-c, 0, a)^t$ and $P(\mathbf{e}_3) = (b, -a, 0)^t$. Furthermore, the matrix of the projection $I + P^2$ is the matrix product $\mathbf{p}\mathbf{p}^t$. Thus the matrix of L is

$$I + (\sin\theta)\begin{bmatrix} 0 & -c & b \\ c & 0 & -a \\ -b & a & 0 \end{bmatrix} + (1 - \cos\theta)\begin{bmatrix} a^2 - 1 & ab & ac \\ ab & b^2 - 1 & bc \\ ac & bc & c^2 - 1 \end{bmatrix}.$$