Clearly, if A and B are $n \times n$ matrices, then $A = \sum_{i=1-n}^{n-1} \alpha_i$ and $B = \sum_{i=1-n}^{n-1} \beta_i$, where $\alpha_i, \beta_i \in \delta_i$. As matrix multiplication is distributive, applying our main fact gives us

$$AB = \sum_{i,j=1-n}^{n-1} \alpha_i \beta_j = \sum_{k=1-n}^{n-1} \gamma_k, \quad \text{where} \quad \gamma_k = \sum_{i+j=k} \alpha_i \beta_j \in \delta_k.$$

To see the benefits of thinking of a matrix as a sum of diagonal shift matrices, consider the usual computational proof that the product of two upper triangular matrices is upper triangular. The result is obscured by the difficulty of picturing all possible row-by-column products. However, if A and B are upper triangular, then A and B are sums of upward shifts only; that is, $A = \sum_{i=0}^{n-1} \alpha_i$ and $B = \sum_{j=0}^{n-1} \beta_j$. From above, the shifts γ_k that occur in the product AB must also be upward, since $k = i + j \geq 0$.

Here are two exercises, taken from Gene Golub and Charles Van Loan, *Matrix Computations* (Johns Hopkins University Press, Baltimore, 1983). Can you write the solutions using both the traditional notation and our diagonal notation?

Problem 1. Show that a strictly upper triangular matrix is nilpotent.

Problem 2. Recall that an $n \times n$ matrix is said to have *upper Hessenberg form* if all of the entries below its subdiagonal are zero. Show that if $A \in \mathbb{R}^{n \times n}$ is upper triangular and $B \in \mathbb{R}^{n \times n}$ is upper Hessenberg, then $C = AB$ is upper Hessenberg.

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Finding a Determinant and Inverse Matrix by Bordering

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Consider the class of $n \times n$ matrices B of the form

$$\begin{bmatrix} b_1 & a_1 & a_1 & \cdots & a_1 \\ a_2 & b_2 & a_2 & \cdots & a_2 \\ a_3 & a_3 & b_3 & \cdots & a_3 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_n & a_n & a_n & \cdots & b_n \end{bmatrix},$$

where for all i, $a_i \neq b_i$. We wish to find convenient formulas for the determinant and, when appropriate, the inverse of such a matrix. There are, of course, all-purpose methods for solving this problem, but here is an approach that takes advantage of the special nature of these matrices and leads to an elegant result.

Lemma. *If B is any invertible matrix and $A = \begin{bmatrix} I & J \\ 0 & B \end{bmatrix}$, where I represents an identity matrix and J is any matrix of appropriate size, then A is invertible and* $A^{-1} = \begin{bmatrix} I & -JB^{-1} \\ 0 & B^{-1} \end{bmatrix}$.

The proof follows from the multiplication of block-partitioned matrices; see David Lay, *Linear Algebra and Its Applications,* (2nd ed., Addison-Wesley, Reading, MA, 1997, pp. 126–127).

We border B with $I = [1]$ and $J = [1, 1, \ldots, 1]$ and reduce the inner $n \times n$ block of A to diagonal form by subtracting the appropriate multiples of row 1 from the other rows; this is possible because of the special nature of the matrix B. These operations leave the determinant unchanged and result in the following matrix:

$$\begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ -a_1 & b_1 - a_1 & 0 & \cdots & 0 \\ -a_2 & 0 & b_2 - a_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ -a_n & 0 & 0 & \cdots & b_n - a_n \end{bmatrix}.$$

Now divide each of the last n rows by its (nonzero) diagonal entry to reduce the original $n \times n$ block to the identity matrix. Each of these steps changes the determinant by the factor of $1/(b_i - a_i)$. Subtracting each of the latter rows from the first then produces a first row $[1 + s, 0, \ldots, 0]$, where $s = \sum_{i=1}^{n} a_i/(b_i - a_i)$. Since the latter row operations leave the determinant unchanged and result in a lower triangular matrix, we find that

$$\det A = \det B = (1 + s) \prod_{i=1}^{n} (b_i - a_i).$$

Clearly B is invertible if and only if $1 + s \neq 0$, and we now add this assumption. Finally, dividing the first row by $1 + s$ gives $[1, 0, \ldots, 0]$, which can then be used to eliminate the nonzero entries in column 1.

When all of the above row operations are applied to the $(n+1) \times 2(n+1)$ matrix $[A : I]$, they lead to $[I : A^{-1}]$, where

$$A^{-1} = \begin{bmatrix} 1 & \dfrac{-1}{(b_1 - a_1)(1+s)} & \cdots & \dfrac{-1}{(b_n - a_n)(1+s)} \\ 0 & \dfrac{1}{b_1 - a_1} - \dfrac{a_1}{(b_1 - a_1)^2(1+s)} & \cdots & \dfrac{-a_1}{(b_1 - a_1)(b_n - a_n)(1+s)} \\ 0 & \dfrac{-a_2}{(b_2 - a_2)(b_1 - a_1)(1+s)} & \cdots & \dfrac{-a_2}{(b_2 - a_2)(b_n - a_n)(1+s)} \\ \vdots & \vdots & \vdots & \vdots \\ 0 & \dfrac{-a_n}{(b_n - a_n)(b_1 - a_1)(1+s)} & \cdots & \dfrac{1}{b_n - a_n} - \dfrac{a_n}{(b_n - a_n)^2(1+s)} \end{bmatrix}.$$

The bordering method yields B^{-1} in a form that appears quite different from the result produced by other inversion methods. The reader may verify this by working out the results of the 2×2 case with and without bordering.

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