It follows, therefore, that

$$A_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix},$$

which is the rotation matrix mentioned in the introduction. So we have established the rotation property of rotation matrices.

Trigonometric consequences. Now that we have the desired rotation property we can use it to derive the familiar angle addition formulas for the sine and cosine functions.

Theorem. *$\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$ and $\sin(\alpha + \beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$.*

Proof. Let $\mathbf{v} = \begin{bmatrix} \cos\beta \\ \sin\beta \end{bmatrix}$. Then $A_\alpha\mathbf{v}$ is the result of rotating $\mathbf{v}$ by α, and therefore

$$A_\alpha\mathbf{v} = \begin{bmatrix} \cos(\alpha+\beta) \\ \sin(\alpha+\beta) \end{bmatrix}.$$

Furthermore,

$$\begin{aligned} A_\alpha\mathbf{v} &= \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}\begin{bmatrix} \cos\beta \\ \sin\beta \end{bmatrix} \\ &= \begin{bmatrix} \cos\alpha\cos\beta - \sin\alpha\sin\beta \\ \sin\alpha\cos\beta + \cos\alpha\sin\beta \end{bmatrix}. \end{aligned}$$

Thus

$$\begin{bmatrix} \cos(\alpha+\beta) \\ \sin(\alpha+\beta) \end{bmatrix} = \begin{bmatrix} \cos\alpha\cos\beta - \sin\alpha\sin\beta \\ \sin\alpha\cos\beta + \cos\alpha\sin\beta \end{bmatrix}.$$

By comparing the corresponding components of the vectors in this last equation we have the desired results.

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A Geometric Interpretation of the Columns of the (Pseudo)Inverse of A

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This capsule describes how the columns of the (pseudo)inverse of a matrix A can be used to provide useful geometric information about the rows of A. Specifically, it shows how the ith column of the (pseudo)inverse of A can be used to project the ith row of A on the span of the other rows (see Figure 1). We begin with an elementary proof of the important special case for which the row space of A spans all of Euclidean n-space E_n.

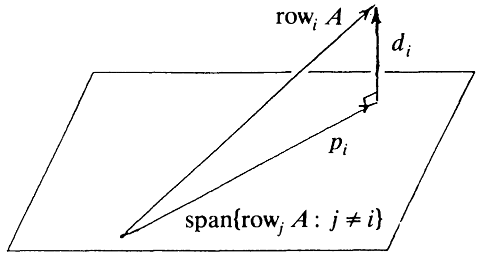

Figure 1

Theorem 1. *Let A be a nonsingular $n \times n$ matrix. Embed the rows of A in E_n, viewed as column vectors, via the natural identification*

$$row_i A \mapsto \mathbf{a}_i = (row_i A)^T, \text{ where } (\ )^T \text{ denotes transpose.}$$

Then for any $i = 1, 2, \ldots, n$, the projection $\mathbf{p}_i$ of $\mathbf{a}_i$ on span $\{\mathbf{a}_j\colon j \neq i\}$ is given by

$$\mathbf{p}_i = \mathbf{a}_i - \mathbf{d}_i, \text{ where } \mathbf{d}_i = \frac{1}{\|col_i A^{-1}\|^2} col_i A^{-1}.$$

Thus, the distance (along $col_i A^{-1}$) from $\mathbf{a}_i$ to span$\{\mathbf{a}_j\colon j \neq i\}$ is $\|\mathbf{d}_i\| = 1/\|col_i A^{-1}\|$.

Proof. If we let $\boldsymbol{\alpha}_j$ denote the jth column of A^{-1} for $j = 1, 2, \ldots, n$, then both $\{\mathbf{a}_i\}_{i=1}^n$ and $\{\boldsymbol{\alpha}_j\}_{j=1}^n$ are bases for E_n. In fact, they are dual bases to each other under the Euclidean inner product $\langle \cdot, \cdot \rangle$ because, since $AA^{-1} = I$,

$$\langle \mathbf{a}_i, \boldsymbol{\alpha}_j \rangle = \delta_{ij}, \text{ where } \delta_{ij} \text{ is the Kronecker delta function.} \tag{1}$$

Fix any i, $1 \le i \le n$. By (1), $\boldsymbol{\alpha}_i$ is orthogonal to the subspace

$$S_i = \text{span}\{\mathbf{a}_j\colon j \neq i\}.$$

So the n linearly independent vectors in $S_i \cup \{\boldsymbol{\alpha}_i\}$ are a basis for the row space of A. Moreover, any $\mathbf{x}$ in the row space of A has the unique representation

$$\mathbf{x} = \mathbf{s} + \gamma\boldsymbol{\alpha}_i, \text{ where } \mathbf{s} \in S_i \text{ and } \gamma = \frac{\langle \mathbf{x}, \boldsymbol{\alpha}_i \rangle}{\|\boldsymbol{\alpha}_i\|^2}$$

[**1**, Theorem 5.3.5, p. 240]. In particular, since $\langle \mathbf{a}_i, \boldsymbol{\alpha}_i \rangle = 1$ by (1), we have

$$\mathbf{a}_i = \mathbf{p}_i + \mathbf{d}_i, \text{ where } \mathbf{p}_i \in S_i \text{ and } \mathbf{d}_i = \frac{1}{\|\boldsymbol{\alpha}_i\|^2}\boldsymbol{\alpha}_i, \tag{2}$$

which is what we set out to prove.

The vectors $\mathbf{p}_i$ and $\mathbf{d}_i$ are, respectively, the components of $\mathbf{a}_i$ on and orthogonal to S_i (see Figure 1). So the relative lengths $\|\mathbf{p}_i\|$, $\|\mathbf{d}_i\|$, and $\|\mathbf{a}_i\|$ can be used to assess the extent to which row$_i A$ is linearly dependent on the other rows of A. To illustrate this, let us project the (transposed) second row of the nonsingular matrix

$$A = \begin{bmatrix} 0 & 2 & 4 \\ 2 & 2 & 9 \\ 1 & -1 & 0 \end{bmatrix}$$

on S_2 = the span of the (transposed) first and third rows. To get $\mathbf{d}_2$, we first solve $A\boldsymbol{\alpha} = \mathbf{e}_2$ for $\boldsymbol{\alpha}_2 = \text{col}_2 A^{-1} = [-2 \quad -2 \quad 1]^T$ and then use (2):

$$\mathbf{d}_2 = \frac{1}{\|\text{col}_2 A^{-1}\|^2}\text{col}_2 A^{-1} = \frac{1}{9}\begin{bmatrix} -2 \\ -2 \\ 1 \end{bmatrix}; \text{ so } \|\mathbf{d}_2\| = \frac{1}{3}.$$

The desired projection of $\mathbf{a}_2 = (\text{row}_2 A)^T$ on S_2 is then easily obtained as

$$\mathbf{p}_2 = \mathbf{a}_2 - \mathbf{d}_2 = \begin{bmatrix} 2 \\ 2 \\ 9 \end{bmatrix} - \frac{1}{9}\begin{bmatrix} -2 \\ -2 \\ 1 \end{bmatrix} = \frac{20}{9}\begin{bmatrix} 1 \\ 1 \\ 4 \end{bmatrix}; \text{ so } \|\mathbf{p}_2\| = \frac{20}{3}\sqrt{2}.$$

Since $\|\mathbf{d}_2\|$ is small compared to $\|\mathbf{a}_2\| = \sqrt{89}$ (or, equivalently, since $\|\mathbf{d}_2\|$ and $\|\mathbf{a}_2\|$ are both 9.43 to two decimal places), we see that row 2 of A is nearly linearly dependent on rows 1 and 3.

If a matrix A is "nearly singular" in this sense, then it is likely to be ill conditioned, that is, small errors in A or $\mathbf{b}$ can produce disproportionately large errors in the solution of the linear system $A\mathbf{x} = \mathbf{b}$. Theorem 1 can thus provide useful insight when examining errors that occur in solving linear $n \times n$ systems.

The proof of Theorem 1 shows that the result is an immediate consequence of (1). This suggests that Theorem 1 can be generalized to matrices A having a right inverse. In fact, the result does generalize to $m \times n$ matrices A if one replaces A^{-1} by the $n \times m$ pseudoinverse of A, which we denote by A^+. A highly readable account of pseudoinverses is given in Appendix A of **[2]**. The discussion there shows the following:

Property 1. For any A, the columns of A^+ lie in the row space of A.

Property 2. If A is $m \times n$ and has rank m, then $AA^+ = I_m$.

Consequently, with only minor notational changes, the proof given for Theorem 1 can be modified to prove the following more general result.

Theorem 2. *Let A be an $m \times n$ matrix of rank m and let $\mathbf{a}_i$ denote $(\text{row}_i A)^T$. Then for any $i = 1, 2, \ldots, m$, the projection $\mathbf{p}_i$ of $\mathbf{a}_i$ on $S_i = \text{span}\{\mathbf{a}_j\colon j \neq i\}$ is given by*

$$\mathbf{p}_i = \mathbf{a}_i - \mathbf{d}_i, \text{ where } \mathbf{d}_i = \frac{1}{\|\text{col}_i A^+\|^2}\text{col}_i A^+$$

where A^+ denotes the pseudoinverse of A.

Theorem 2 can provide useful insight when performing factor analysis or in other situations where one might want to know if a vector in Euclidean space "nearly" lies in the span of a set of vectors that may not be a basis.

References

1. H. Anton, *Elementary Linear Algebra*, 6th ed., John Wiley, NY, 1991.
2. G. Strang, *Linear Algebra and Its Applications*, 3rd ed., Harcourt, Brace, Jovanovich, San Diego, 1988.