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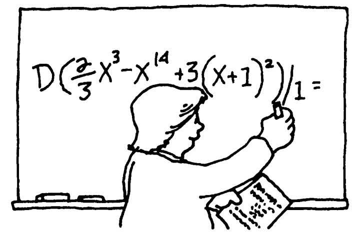

A Classroom Capsule is a short article that contains a new insight on a topic taught in the earlier years of undergraduate mathematics. Please submit manuscripts prepared according to the guidelines on the inside front cover to Tom Farmer.

Generating Exotic-Looking Vector Spaces

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Many introductory textbooks on linear algebra introduce the definition of a vector space using *abstract* notation for vector addition and scalar multiplication (such as $\oplus$ and $\odot$, respectively), but then they generally limit the exercises and examples to classic problems where vector addition and scalar multiplication are those defined in $\mathbb{R}^n$. This note describes how to generate *computational exercises* designed for teaching students the axioms of vector spaces using nonstandard operations for vector addition and scalar multiplication. Such exercises have the pedagogical value of allowing the student to study the axioms of vector spaces using familiar objects, such as real numbers, but with unfamiliar operations for vector addition and scalar multiplication. Checking the vector space axioms in such exotic vector spaces helps students develop a deeper understanding of these axioms. The basis for generating these exercises lies in the following theorem.

Theorem. *Let $\mathbb{R}$ be the field of real numbers and let $f : \mathbb{R} \to V$ be a one-to-one function from $\mathbb{R}$ onto a codomain V. If we define vector addition by*

$$x \oplus y = f\left(f^{-1}(x) + f^{-1}(y)\right) \tag{1}$$

and scalar multiplication by

$$\alpha \odot x = f\left(\alpha \cdot f^{-1}(x)\right) \tag{2}$$

for all x and y in V and all α in $\mathbb{R}$, then the set V, together with the operations $\oplus$ and $\odot$, form a vector space over the field of real numbers.

Here, the operations $+$ and $\cdot$ are ordinary addition and multiplication of real numbers. The additive identity for the vector space is $0 \odot x = f(0)$, and the additive inverse for the element x in V is $(-1) \odot x = f\left(-f^{-1}(x)\right)$. The real vector space V is sometimes denoted more formally by $(V, \oplus, \odot)$.

Proof. We need only note that when f^{-1} is applied to both sides of equations (1) and (2), we get

$$f^{-1}(x \oplus y) = f^{-1}(x) + f^{-1}(y) \quad \text{and} \quad f^{-1}(\alpha \odot x) = \alpha \cdot f^{-1}(x),$$

respectively, showing that f^{-1} is an isomorphism of $(V, \oplus, \odot)$ onto $(\mathbb{R}, +, \cdot)$, which is the vector space of real numbers under the usual operations of addition and scalar multiplication. □

For a more direct algebraic proof, we can test the individual vector space axioms. For example, to establish the axiom $\alpha \odot (\beta \odot x) = (\alpha \cdot \beta) \odot x$, we write

$$\begin{aligned}\alpha \odot (\beta \odot x) &= \alpha \odot f\left(\beta \cdot f^{-1}(x)\right) = f\left(\alpha \cdot f^{-1}\left(f\left(\beta \cdot f^{-1}(x)\right)\right)\right)\\ &= f\left(\alpha \cdot \left(\beta \cdot f^{-1}(x)\right)\right) = f\left((\alpha \cdot \beta) \cdot f^{-1}(x)\right) = (\alpha \cdot \beta) \odot x.\end{aligned}$$

The following examples show how this theorem may be used to generate exercises in the study of vector spaces.

Exercise 1. Let β be any positive real number and let $f : \mathbb{R} \to \mathbb{R}^+$ be defined by $f(x) = (1/\beta)e^x$. Then f is a one-to-one function from $\mathbb{R}$ onto the set of positive real numbers, and $f^{-1}(x) = \ln(\beta x)$ for $x > 0$. Using equations (1) and (2), we would define vector addition and scalar multiplication by

$$x \oplus y = \frac{1}{\beta} e^{\ln(\beta x) + \ln(\beta y)} = \beta x y \quad \text{and} \quad \alpha \odot x = \frac{1}{\beta} e^{\alpha \ln(\beta x)} = \beta^{\alpha - 1} x^{\alpha},$$

respectively. Show that for any $\beta > 0$, the set of positive real numbers together with the operations $x \oplus y = \beta x y$ and $\alpha \odot x = \beta^{\alpha-1} x^{\alpha}$ form a vector space over the field of real numbers.

When $\beta = 1$ the operations in exercise 1 simplify to $x \oplus y = xy$ and $\alpha \odot x = x^{\alpha}$, an example provided in many linear algebra textbooks.

Exercise 2. Let n be an odd positive integer and define $f : \mathbb{R} \to \mathbb{R}$ so that $f(x)$ is the principal value of $x^{1/n}$. Then f is a one-to-one function from $\mathbb{R}$ onto $\mathbb{R}$ and $f^{-1}(x) = x^n$. Therefore, we would define vector addition and scalar multiplication by

$$x \oplus y = (x^n + y^n)^{1/n} \quad \text{and} \quad \alpha \odot x = (\alpha x^n)^{1/n} = \alpha^{1/n} x.$$

Show that when n is an odd positive integer, the set of all real numbers together with the operations $x \oplus y = (x^n + y^n)^{1/n}$ and $\alpha \odot x = \alpha^{1/n} x$ form a vector space over the field of real numbers.

Exercise 3. Suppose that b is any real number and $f(x) = x + b$. Then f is a one-to-one function from $\mathbb{R}$ onto $\mathbb{R}$ having $f^{-1}(x) = x - b$. Therefore, we would define vector addition and scalar multiplication by

$$x \oplus y = (x - b) + (y - b) + b = x + y - b \quad \text{and} \quad \alpha \odot x = \alpha(x - b) + b.$$

Show that for any real number b, the set of all real numbers together with the operations $x \oplus y = x + y - b$ and $\alpha \odot x = \alpha x + b(1 - \alpha)$ form a vector space over the field of real numbers.

Exercise 4. Let $f : \mathbb{R} \to (-1, 1)$ be defined by $f(x) = \tanh x$. Then f is a one-to-one function from $\mathbb{R}$ onto the set of real numbers in the open interval $(-1, 1)$, and

$$f^{-1}(x) = \tanh^{-1} x = \frac{1}{2} \ln \left(\frac{1+x}{1-x} \right)$$

for $-1 < x < 1$. Vector addition and scalar multiplication in the theorem become

$$x \oplus y = \tanh\left(\tanh^{-1} x + \tanh^{-1} y\right) \quad \text{and} \quad \alpha \odot x = \tanh\left(\alpha \tanh^{-1} x\right),$$

which reduce to

$$x \oplus y = \frac{x+y}{1+xy} \quad \text{and} \quad \alpha \odot x = \frac{(1+x)^\alpha - (1-x)^\alpha}{(1+x)^\alpha + (1-x)^\alpha} \tag{3}$$

respectively. Show that the set of real numbers in the interval $(-1, 1)$, together with the operations in equation (3), form a vector space over the field of real numbers.

After your students have worked this exercise, you may want to ask them why the set of real numbers in the smaller interval $(-\frac{1}{2}, \frac{1}{2})$ together with the operations in equation (3) *do not* form a vector space over the field of real numbers. In answering this question they learn to appreciate the requirement of closure.

Other applications. Some students may wonder why we bother to define vector addition and scalar multiplication in an abstract way. Where is the application in this? One response to this question lies in exercise 4, which has a direct application in Einstein's special theory of relativity. In one-dimensional special relativity, velocities x and y (whose magnitudes are given as fractions of the speed of light) do not add in the usual way [see Richard Mould, *Basic Relativity*, Springer-Verlag, New York, 1994, page 36]. Rather, they add according to the rule

$$x \oplus y = \frac{x+y}{1+xy}.$$

Using this observation, the following problem in special relativity becomes easy to solve.

> Suppose in a galactic rocket convoy, n rockets are traveling from Earth to Alpha Centauri, the nearest star to our sun. Assume that earthbound observers see rocket 1 (the slowest and closest) receding at a velocity v. Each pilot, who is watching the rocket directly ahead in the convoy, sees this rocket receding at the velocity v as well. What is the velocity of the leading rocket, as measured by observers on earth?

To determine this, we need only note that the vector addition on $(-1, 1)$ for v is precisely the special relativistic velocity addition formula, so the desired velocity is

$$v \oplus v \oplus v \oplus \cdots \oplus v = n \odot v = \frac{(1+v)^n - (1-v)^n}{(1+v)^n + (1-v)^n}.$$

The number of exercises that can be constructed using equations (1) and (2) is limitless. Just choose your favorite one-to-one function from $\mathbb{R}$ onto a codomain V and use both equations in defining vector addition and scalar multiplication. *Of course, your students should be asked to do each of these exercises using the vector space axioms, without knowledge of the theorem.* In this way when students demonstrate the property $(\alpha+\beta)\odot x = (\alpha\odot x)\oplus(\beta\odot x)$ in each of these exercises, they will better understand that $\alpha+\beta$ is just the ordinary addition of the real numbers α and β in the field $\mathbb{R}$, while $(\alpha\odot x)\oplus(\beta\odot x)$ is the vector addition of the elements $\alpha\odot x$ and $\beta\odot x$ in V. Similarly, when students demonstrate the property $\alpha\odot(\beta\odot x) = (a\cdot\beta)\odot x$, they will better understand that $\alpha\cdot\beta$ is just the ordinary multiplication of the real numbers α and β in the field $\mathbb{R}$, while $\alpha\odot(\beta\odot x)$ involves the scalar multiplication of elements in the field $\mathbb{R}$ with elements in the vector space V.

All of the vector spaces constructed using the above theorem are one-dimensional. Exotic higher-dimensional vector spaces can be formed using a *direct sum* of such one-dimensional vector spaces. These higher-dimensional vector spaces can then be used to construct some very interesting exercises.

For example, using the direct sum of the vector spaces in exercise 1 (with $\beta = 1$), exercise 2 (with $n = 3$), and exercise 3 (with $b = -1$), we can let V be the set of 3×1 matrices

$$V = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \middle| \quad x, y, z \in \mathbb{R} \quad \text{with} \quad x > 0 \right\},$$

with vector addition defined by

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \oplus \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} xx' \\ (y^3 + y'^3)^{1/3} \\ z + z' + 1 \end{pmatrix}$$

and scalar multiplication defined by

$$\alpha \odot \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x^\alpha \\ \alpha^{1/3} y \\ \alpha z + \alpha - 1 \end{pmatrix}.$$

Then V with these operations forms a three-dimensional vector space over the field of real numbers $\mathbb{R}$. The additive identity for this vector space is the matrix

$$0 \odot \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix},$$

and the additive inverse of $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ is

$$(-1) \odot \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1/x \\ -y \\ -z - 2 \end{pmatrix}.$$

Once these higher-dimensional vector spaces are introduced, it is easy to generate many challenging problems involving the concepts of basis sets, coordinate vectors, inner products, and linear transformations.

With the theorem as a guide, an instructor of linear algebra can construct interesting and fun exercises that test understanding of the abstract nature of vector spaces. If your students can solve the preceding exercises, you can be confident that they have grasped the concepts behind the vector space axioms. I have used similar exercises in my classes for the last six years, and many of my students have found them challenging, instructive, and fun to do.

Equation (1) can also be used to generate examples of exotic-looking abelian groups. Furthermore, by combining equation (1) with $x \otimes y = f\left(f^{-1}(x) \cdot f^{-1}(y)\right)$, you can generate exotic-looking fields for beginning abstract algebra students. Later, when the concept of isomorphism is introduced, it is nice to return to these examples and have students find an isomorphism between the exotic structure and the familiar structure on $\mathbb{R}$.

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Nothing Counts for Something

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Good examples can bring a course to life and they are memorable long afterward. This brief note discusses a simple problem appropriate to any course that develops elementary methods of discrete counting. It involves the old-fashioned floor lamp with numerous bulbs and multiple switches, for which a natural question is, how many levels of light are possible? Students can relate to this puzzle, for some have seen these lamps and it's possible to bring one to class for a rare, live demonstration of the subject of study.

The problem with nothing. The history of number systems shows that the introduction of zero was a major advance [**1–3**], both for its role as a place holder in number representations and also as a number itself that can be used in ordinary arithmetic and algebraic computations. The difficulty with the idea that nothing can be something persists among today's students, for whom the empty set or vacuous state is often overlooked. This is evident whenever my example of a lamp is used in teaching elementary combinatorics.

I pose the problem by inviting my class to consider the number of illumination levels available in a floor lamp having the following features. Light is provided by one large, central bulb together with three smaller bulbs clustered around the main stem. The central bulb can shine with, say, 50, 100, or 150 watts depending on its switch setting. A second switch also rotates among four settings, one illuminating a single outer bulb, the next powering instead the other two smaller bulbs, and the third lighting up all three outer bulbs. Let's suppose that each of these three is a 60-watt bulb, in order to avoid overlap in the wattage output of different combinations.

It has been my experience that when students are asked for the number of possible light levels, they respond with a dazzling variety of proposed counts. Nine, as the product of three outer levels with three central bulb levels, is a common response. Classes over the years have been creative in thinking up other counts as well, and they generally ignore the fact that the "off" position for each switch needs to be considered. Including "off," there are four levels for each of the two switches, and the multiplication principle leads immediately to 16 levels of illumination. Occasionally a bright or experienced student comes up with the correct enumeration. Even more unusual a response is 15, representing the number of nontrivial levels of illumination.