(assuming $k \neq 0$). Hence, a critical point t_0 provides a local minimum (maximum) if the distance from q to P_0 is less (greater) than the radius of curvature at P_0.

A graphical approach becomes apparent when we introduce the *evolute* of C. Let r be the radius of curvature of C at a point P, and consider the normal line to C at P, drawn to the concave side. Recall, the center of curvature of C corresponding to P is the point Q on this normal whose distance from P is r. The evolute E is the locus of points generated by Q as P moves along C. The situation is depicted in Figure 2.

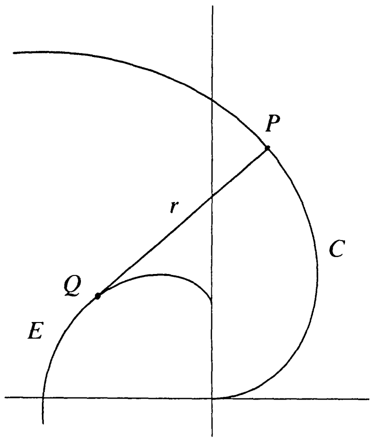

Figure 2

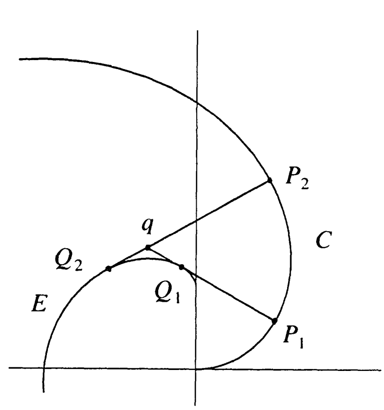

Figure 3

Armed with the evolute and the theorem above, we can visually classify the critical points of f. Let P_1 and P_2 be the points on C corresponding to t_1 and t_2. In Figure 3, we see that f has a local maximum at t_1 and a local minimum at t_2. This is because the distance from q to P_1 is greater than the distance from Q_1 to P_1, the latter being the radius of curvature at P_1, and the distance from q to P_2 is less than the distance from Q_2 to P_2, the latter being the radius of curvature at P_2.

A property of evolutes suggested by Figure 3 is that the normal line to C at any point (drawn to the concave side) is tangent to the evolute. Simmons proves this on pp. 749–750. In principle, this provides a complete graphical determination (location and classification) of the critical points of f by sketching all possible tangent lines to E passing through q. Each such tangent line is normal to C, the point of intersection with C determining a critical point of f.

Finally, for interested readers, the curve C in the figures is the spiral $\mathbf{r}(t) = (2t\cos t, 2t\sin t)$. The evolute is easy to calculate and graph with a symbolic-graphing package.

Parametric Equations and Planar Curves

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One of the most satisfying fringe benefits of teaching mathematics is seeing unexpected and intriguing connections between various branches of mathematics while attempting to help others learn. In challenging students of multivariate calculus with problem solving situations outside the ordinary, we devised the following problem: *Determine whether the three-dimensional curve represented parametrically by system* (1) *is planar*, *i.e.*, *is contained in a plane*.

$$x = t^2 + 1$$
$$y = 4t^2 - 2t - 1$$
$$z = t^2 - t \tag{1}$$

We wanted to give the students a parametric representation involving quadratics in t such that the answer to the question was affirmative. (If not, one would need only to find four points of the curve and show they were noncoplanar.) But in considering several different examples, we were surprised to find that each one represented a planar curve. The obvious conjecture leads to the following algebraic–geometric result, which in turn raises questions whose answers exemplify the interplay between elementary vector geometry and linear algebra: *The curve represented parametrically by system* (2) *is planar*.

$$x(t) = a_1t^2 + b_1t + c_1$$
$$y(t) = a_2t^2 + b_2t + c_2$$
$$z(t) = a_3t^2 + b_3t + c_3. \tag{2}$$

For justification, let t_1, t_2, and t_3 be three values of t that yield three noncollinear points on the curve. (If all points are collinear, the result is obviously true.) Let $Ax + By + Cz = D$ be the unique plane containing the three points thus generated. Consider the equation $Ax(t) + By(t) + Cz(t) = D$, or

$$A\left(a_1t^2 + b_1t + c_1\right) + B\left(a_2t^2 + b_2t + c_2\right) + C\left(a_3t^2 + b_3t + c_3\right) = D. \tag{3}$$

This is an equation in t of degree at most two and having three solutions t_1, t_2, and t_3. Since a quadratic equation has at most two solutions, (3) must be an identity; that is, *every* value of t satisfies (3). Thus, the curve represented by (2) lies entirely in the plane $Ax + By + Cz = D$. (Note that this result can be obtained quickly using more advanced methods of vector calculus; since the third derivatives of the equations in (2) are each 0, the torsion of the curve is 0, implying that the curve is planar. Our purpose is to use means accessible to most calculus students.)

Of course, the next question is naturally "Which planar curve is represented by (2)?" It turns out that a planar curve of the form (2) is either a parabola or a degenerate case (a line, a ray, or a point). As an illustrative example, in lieu of the cumbersome general case, consider the parametric equations (1). This curve lies in the plane $2x - y + 2z = 3$. A vector normal to this plane is $2\mathbf{i} - \mathbf{j} + 2\mathbf{k}$. Extend this vector to an orthogonal basis for $\mathbf{R}^3$, say $\{2\mathbf{i} + 2\mathbf{j} - \mathbf{k},\ \mathbf{i} - 2\mathbf{j} - 2\mathbf{k},\ 2\mathbf{i} - \mathbf{j} + 2\mathbf{k}\}$. (Note that the first and second vectors are thus parallel to the plane containing the given curve.) The corresponding orthonormal basis for $\mathbf{R}^3$ is $\{\frac{2}{3}\mathbf{i} + \frac{2}{3}\mathbf{j} - \frac{1}{3}\mathbf{k},\ \frac{1}{3}\mathbf{i} - \frac{2}{3}\mathbf{j} - \frac{2}{3}\mathbf{k},\ \frac{2}{3}\mathbf{i} - \frac{1}{3}\mathbf{j} + \frac{2}{3}\mathbf{k}\}$. The associated orthogonal matrix is

$$\mathbf{K} = \begin{bmatrix} \frac{2}{3} & \frac{2}{3} & -\frac{1}{3} \\ \frac{1}{3} & -\frac{2}{3} & -\frac{2}{3} \\ \frac{2}{3} & -\frac{1}{3} & \frac{2}{3} \end{bmatrix}.$$

Curve (1) may be written in matrix form

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ 4 & -2 & -1 \\ 1 & -1 & 0 \end{bmatrix} \cdot \begin{bmatrix} t^2 \\ t \\ 1 \end{bmatrix}.$$

Multiplying both sides of this equation by the orthogonal matrix **K**, we obtain

$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \mathbf{K} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 & -1 & 0 \\ -3 & 2 & 1 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} t^2 \\ t \\ 1 \end{bmatrix},$$

so that, in the new $x'y'z'$ coordinate system, the curve (1) has parametric equations

$$\begin{aligned} x' &= 3t^2 - t \\ y' &= -3t^2 + 2t + 1 \\ z' &= 1 \end{aligned}$$

Similarly, with respect to a new $x'y'z'$ coordinate system, the general curve (2) has parametric equations of the form

$$\begin{aligned} x' &= a_1' t^2 + b_1' t + c_1' \\ y' &= a_2' t^2 + b_2' t + c_2' \\ z' &= c_3'. \end{aligned}$$

So the identification of our curve is reduced to that of the curve parameterized as

$$\begin{aligned} x' &= a_1' t^2 + b_1' t + c_1' \\ y' &= a_2' t^2 + b_2' t + c_2' \end{aligned} \tag{4}$$

lying in the plane $z' = c_3'$. We can accomplish this identification through either of two equivalent methods, one algebraic and one geometric.

Method 1 (*algebraic*). Adding suitable multiples of the two equations, we can eliminate the t^2 terms to obtain a linear expression for t in terms of x' and y'. Substituting this expression into the first equation in (4) yields a quadratic in x' and y' that has discriminant 0, so that our curve is a parabola or one of its degenerate cases.

Method 2 (*geometric*). By a suitable rotation of the $x'y'$ axes parallel to this plane, we can put the planar curve (4) in the form

$$\begin{aligned} x'' &= a_1'' t^2 + b_1'' t + c_1'' \\ y'' &= \phantom{a_1'' t^2 + {}} b_2'' t + c_2''. \end{aligned}$$

(Geometrically, the required rotation above is a counterclockwise rotation through an angle of θ, where $\tan\theta = a_2'/a_1'$.) We can now eliminate t easily to show that our curve is indeed a parabola, or one of its degenerate cases.