CLASSROOM CAPSULES

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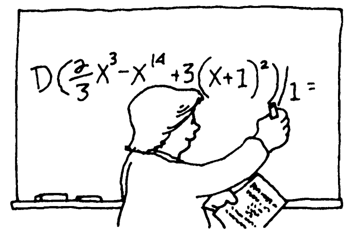

A Classroom Capsule is a short article that contains a new insight on a topic taught in the earlier years of undergraduate mathematics. Please submit manuscripts prepared according to the guidelines on the inside front cover to Tom Farmer.

Eigenvalues of Matrices of Low Rank

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While it can be difficult, or even impossible, to find exact eigenvalues of an $n \times n$ matrix in general, we will illustrate a simple method that works when the rank of the matrix is small. This technique can be used for student discovery in a linear algebra class; an instructor can assign a sequence of exercises requiring students to solve special cases, make conjectures about generalizations, and then prove their *conjectures*.

The eigenvalues of a matrix A are the zeroes of its characteristic polynomial, $\det(\lambda I - A)$, which can be written as

$$P(\lambda) = \lambda^n - c_{n-1}\lambda^{n-1} + c_{n-2}\lambda^{n-2} - \dots + (-1)^n c_0. \tag{1}$$

It is well known (see, for example, [**3**, p. 251]) that the coefficients c_{n-1} and c_0 in (1) are, respectively, the trace of A (the sum of its diagonal entries) and the determinant of A. In fact, all coefficients in (1) can be expressed in terms of k-rowed principal minors of A. (A *k-rowed principal minor* of an $n \times n$ matrix A is the determinant of a $k \times k$ submatrix of A whose entries, a_{ij}, have indices i and j that are the elements of the same k-element subset of $\{1, 2, \dots, n\}$.)

For example, if

$$B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix},$$

then:

the 1-rowed principal minors of B are the diagonal entries $1, 5, 9$,
the 2-rowed principal minors of B are

$$\det\begin{bmatrix} 1 & 2 \\ 4 & 5 \end{bmatrix} = -3, \quad \det\begin{bmatrix} 5 & 6 \\ 8 & 9 \end{bmatrix} = -3, \quad \det\begin{bmatrix} 1 & 3 \\ 7 & 9 \end{bmatrix} = -12,$$

and the only 3-rowed principal minor of B is $\det(B) = 0$.

The following theorem is the key to the subsequent results. It dates back to 1875 (see [**2**, p. 295]), but is not so well-known today. A proof can be found in [**1**, p. 215].

Theorem 1. *Let A be an $n \times n$ matrix. Then, for $k = 1, 2, \ldots, n$, the coefficient c_{n-k} in the characteristic polynomial* (1) *is given by the sum of the k-rowed principal minors of A.*

For example, the characteristic polynomial of the matrix B of the previous example is $P(\lambda) = \lambda^3 - c_2\lambda^2 + c_1\lambda - c_0$, where $c_2 = 1 + 5 + 9 = 15$, $c_1 = -3 + (-3) + (-12) = -18$, and $c_0 = 0$. Thus, $P(\lambda) = \lambda^3 - 15\lambda^2 - 18\lambda$.

Theorem 1 does not usually provide an efficient way to compute the characteristic polynomial of A; computing $\det(\lambda I - A)$ directly generally requires fewer operations. When the rank of A is small, however, this theorem does provide an efficient means of finding the eigenvalues of A.

It is well known (see, for example, [**3**, p. 202]) that the rank of A is equal to the order of the largest nonzero *minor* of A—the largest square submatrix of A with nonzero determinant. Since a k-rowed principal minor is a special type of minor, the following theorem follows immediately from this result.

Theorem 2. *Let A be an $n \times n$ matrix. Then, the order of the largest nonzero k-rowed principal minor of A is less than or equal to the rank of A.*

It follows from Theorems 1 and 2 that if A is an $n \times n$ matrix and the rank of A is r, then $c_{n-k} = 0$ for $k > r$. Thus, the characteristic polynomial of such a matrix has the form

$$\begin{aligned} P(\lambda) &= \lambda^n - c_{n-1}\lambda^{n-1} + c_{n-2}\lambda^{n-2} - \ldots + (-1)^r c_{n-r}\lambda^{n-r} \\ &= \lambda^{n-r}\left(\lambda^r - c_{n-1}\lambda^{r-1} + c_{n-2}\lambda^{r-2} - \ldots + (-1)^r c_{n-r}\right) \end{aligned}$$

and we obtain the following result.

Corollary. *Let A be an $n \times n$ matrix with rank r. Then, all nonzero eigenvalues of A are among the zeros of the polynomial*

$$Q(\lambda) = \lambda^r - c_{n-1}\lambda^{r-1} + c_{n-2}\lambda^{r-2} - \ldots + (-1)^r c_{n-r}, \tag{2}$$

where c_{n-k}, for $k = 1, 2, \ldots, r$, is given by the sum of the k-rowed principal minors of A.

So, the exact eigenvalues of a matrix of rank 1 or 2 can be found by solving a linear or quadratic equation. For example, the matrix B of the previous examples has rank 2, and its two nonzero eigenvalues are the zeros of $Q(\lambda) = \lambda^2 - 15\lambda - 18$.

The entries of the matrix B, when read from left to right and top to bottom, are those of an arithmetic sequence with common difference equal to 1. Every matrix whose entries form an arithmetic sequence has rank less than or equal to 2. (This can be seen by subtracting the first row from each of the others, leading to a row-reduced matrix with at most two nonzero rows.) Therefore, such matrices have eigenvalues that can be easily found. For example, let

$$H = \begin{bmatrix} 0 & 1 & 2 & 3 \\ 4 & 5 & 6 & 7 \\ 8 & 9 & 10 & 11 \\ 12 & 13 & 14 & 15 \end{bmatrix}.$$

Its entries form an arithmetic sequence, and H has rank 2. So, two of its eigenvalues are equal to 0 and, by the Corollary, the other two satisfy the quadratic equation $\lambda^2 - 30\lambda - 80 = 0$. Thus, the eigenvalues of H are 0, 0, $15 + \sqrt{305}$, and $15 - \sqrt{305}$.

There is an easy way to generate additional examples to which the Corollary can be applied: If the entries of an $n \times n$ matrix A are the terms of a *geometric* sequence, then its rank is 1 (unless all its entries are zero); if the entries of A are consecutive terms of the *Fibonacci* sequence $\{0, 1, 1, 2, 3, 5, 8, \dots\}$, then its rank is 2 for all $n \geq 2$.

The results presented in this article can be used effectively in a linear algebra class, not by presenting them as *faits accomplis*, but instead by having students discover them for themselves. To accomplish this, an instructor can assign a sequence of exercises, varying from the simple to the complex, that ask the student to solve special cases, make conjectures about generalizations, and then prove the conjectures. To be practical, this process requires the help of symbolic mathematics software such as *Mathematica* (but only a very small subset of it), and only the very best students will be able to solve all the exercises.

Here is a typical sequence of exercises that allows students to explore this subject:

(i) Determine the eigenvalues of the $n \times n$ matrix with all entries 1, for $n = 2$, 3, and 4. Make a conjecture for general n. Do the same for a matrix with all entries equal to k. Prove your conjectures. Prove that the rank of all these matrices is one. Make a general statement about the nature of the eigenvalues of an $n \times n$ matrix of rank one. Prove this statement.

(ii) Find the eigenvalues of the 2×2, 3×3, and 4×4 matrices whose entries are the nonnegative integers, listed in order. (The 4×4 case is the matrix H in this article.) In each case, find the characteristic equation for the matrix. Calculate the k-rowed principal minors of these matrices. Express the coefficients in each characteristic equation in terms of sums of these determinants. Make a conjecture about the rank of an $n \times n$ matrix whose entries are the nonnegative integers. Prove this conjecture. What form does the characteristic equation of such a matrix take? How could you use determinants to compute its coefficients?

(iii) Notice that the entries of each matrix in the preceding exercise form an arithmetic sequence. Repeat this exercise for matrices whose entries are those of the arithmetic sequence: $a, a + d, a + 2d, \dots,$ for other values of a and d.

(iv) Again, repeat this exercise, this time for matrices whose entries are those of the Fibonacci sequence: $0, 1, 1, 2, 3, 5, 8, \dots$.

And so on. We have assigned some of these exercises as optional extra-credit homework for students to work individually. However, the material presented in this article might also be used for longer independent or cooperative study projects.

References

1. T. Muir, *Theory of Determinants*, Volumes 3 and 4, Dover, 1960, originally published in 1882.
2. F. Hohn, *Elementary Linear Algebra*, Macmillan, 1958.
3. G. Strang, *Linear Algebra and Its Applications*, 3rd ed., Saunders, 1988.