Approaches to the Formula for the *n*th Fibonacci Number

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In this capsule we advocate proving the same theorem in different courses. This helps the undergraduate view mathematics as *a unified whole with a variety of techniques*.

To illustrate, we review proofs of the equivalence of the two most common definitions of the Fibonacci numbers (cf. [**5**] and [**6**]). There seems to be controversy in the literature as to what the standard definition of the Fibonacci numbers should be, particularly in regard to the initial values. We therefore follow the standard definition of the Fibonacci Association [**2**]. Cogent arguments for using alternate initial values may be found in [**17**]. For a recent text on the Fibonacci numbers see [**15**].

Definition 1. [Fibonacci's Recursion] The Fibonacci numbers are defined recursively for integer $n \geq 0$ by

$$F_0 = 0, \quad F_1 = 1, \quad F_{n+2} = F_{n+1} + F_n. \tag{1}$$

Definition 2. [Binet's Formula] Alternatively we can define the Fibonacci numbers for integer n by

$$F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}, \tag{2}$$

where

$$\alpha = \frac{1+\sqrt{5}}{2} \quad \text{and} \quad \beta = \frac{1-\sqrt{5}}{2}$$

are the two solutions of $p(z) = z^2 - z - 1 = 0$. Note that $\alpha + \beta = 1$ and $\alpha\beta = -1$, facts that will be used in the following proofs.

We shall prove the equivalence of the two definitions, (1) and (2), using the basic methods of five standard undergraduate courses.

Elementary Algebra. First note that $\alpha + 1 = \alpha^2$ and $\beta + 1 = \beta^2$. Then

$$\frac{\alpha^{n+1} - \beta^{n+1}}{\alpha - \beta} + \frac{\alpha^n - \beta^n}{\alpha - \beta} = \frac{\alpha^n(\alpha + 1) - \beta^n(\beta + 1)}{\alpha - \beta} = \frac{\alpha^{n+2} - \beta^{n+2}}{\alpha - \beta},$$

which demonstrates that (2) implies (1).

In the remaining courses we demonstrate that (1) implies (2).

Linear Algebra. Define matrices

$$M = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \quad P = \begin{bmatrix} 1 & 1 \\ \alpha & \beta \end{bmatrix} \quad D = \begin{bmatrix} \alpha & 0 \\ 0 & \beta \end{bmatrix},$$

and for integer $n \geq 1$, define

$$v_n = \begin{bmatrix} F_{n-1} \\ F_n \end{bmatrix}.$$

Standard computations show that

$$P^{-1} = \frac{1}{\beta - \alpha}\begin{bmatrix} \beta & -1 \\ -\alpha & 1 \end{bmatrix},$$

and

$$D^n = \begin{bmatrix} \alpha^n & 0 \\ 0 & \beta^n \end{bmatrix}.$$

Moreover, the characteristic polynomial of M is $p(z)$ and a diagonal decomposition of M is $M = PDP^{-1}$. Then $Mv_n = v_{n+1}$, and two routine inductions show that $v_n = M^{n-1}v_1$ and $M^n = PD^nP^{-1}$. We conclude that $v_n = PD^{n-1}P^{-1}v_1$, which, upon expansion, yields (2) in vector form.

Matrices frequently provide alternate elegant proofs of identities in Fibonacci numbers **[1]**, **[7]**, **[10]**. Several recent undergraduate linear algebra texts that develop the above approach are **[13]**, **[14]**, and **[18]**.

Calculus II. Define the function

$$G(x) = \sum_{0 \le n < \infty} F_n x^n.$$

Using (1) an easy induction shows $F_n \le 2^n$ and consequently $G(x)$ is absolutely convergent for $|x| < 1/2$. Another use of (1) demonstrates that $x(G + xG) = G - x$. This implies

$$G(x) = \frac{x}{1 - x - x^2}.$$

Following **[16]**, **[17]**, or **[3]** we use a partial fraction decomposition and the formula for geometric series. This yields

$$\frac{x}{1 - x - x^2} = \frac{1}{\alpha - \beta}\left(\frac{1}{1 - \alpha x} - \frac{1}{1 - \beta x}\right)$$

$$= \frac{1}{\alpha - \beta}\sum(\alpha^n - \beta^n)x^n$$

and (2) immediately follows. A recent undergraduate text advocating this approach with generalizations is **[9]**.

Number Theory. The method of characteristic equations, which, in the simplest case, finds closed formulae by solving the characteristic equation and forming linear combinations of powers of its roots, is the best known method of deriving (2) from (1). An example of a standard undergraduate text in number theory, the course where this method is traditionally presented, is **[12]**.

Suppose the recursive sequence G_n satisfies the recursion $G_{n+2} = aG_{n+1} + bG_n$. We form the *characteristic polynomial* $z^2 - az - b$ and let r_1 and r_2 be the two roots of $z^2 - az - b = 0$. Then

$$G_n = \frac{(G_1 - r_1 G_0) r_2^n - (G_1 - r_2 G_0) r_1^n}{r_2 - r_1}.$$

In particular if we let $a = 1$, $b = 1$, $G_0 = 0$, $G_1 = 1$, then the characteristic polynomial is $p(z)$ and we derive (2).

The proof presented in [**12**] uses manipulations. A more general approach [**8**] can be developed, for any linear difference equation of order k with constant coefficients, by noting that a sufficient condition for the sequence r^n, r complex, to belong to the complex, k-dimensional, vector space of all sequences satisfying a given difference equation, is that r is a root of the characteristic polynomial. If the characteristic polynomial has k distinct roots then a closed formula for any point in the space can be found by calculating its coordinates relative to the basis of sequences r^n where r varies over the roots of the characteristic polynomial. [**8**] also gives a short but completely detailed development of the relationship between characteristic polynomials, generating functions, and closed formulae for recursions. Another good expository account of these relationships may be found in [**16**].

Complex Variables. Again, consider the complex analytic generating function $G(z) = z/(1 - z - z^2)$. Let C_T be the circle of radius $T > 2$ around the origin. Let $C_{-\alpha}$, $C_{-\beta}$, and C_0 be the circles of radius $1/4$ around $-\alpha$, $-\beta$, and 0 respectively. The residue theorem states that for $n \geq 0$

$$\frac{1}{2\pi i}\int_{C_T} \frac{G(z)}{z^{n+1}} dz = \frac{1}{2\pi i}\left(\int_{C_0} \frac{G(z)}{z^{n+1}}\, dz + \int_{C_{-\alpha}} \frac{G(z)}{z^{n+1}}\, dz + \int_{C_{-\beta}} \frac{G(z)}{z^{n+1}}\, dz\right).$$

By the triangle inequality for integrals, as T goes to ∞,

$$\left|\frac{1}{2\pi i}\int_{C_T} \frac{G(z)}{z^{n+1}}\, dz\right| \leq O(T^{-1}) \to 0.$$

By the Cauchy integral formula for derivatives,

$$\frac{1}{2\pi i}\int_{C_0} \frac{G(z)}{z^{n+1}}\, dz = \frac{G^{(n)}(0)}{n!} = F_n.$$

Finally, by the Cauchy integral formula we have

$$\frac{1}{2\pi i}\int_{C_{-\alpha}} \frac{G(z)}{z^{n+1}}\, dz = \text{Residue at } -\alpha = \frac{\alpha(-1)^{n+1}}{(\beta - \alpha)\alpha^{n+1}} \quad \text{and}$$

$$\frac{1}{2\pi i}\int_{C_{-\beta}} \frac{G(z)}{z^{n+1}}\, dz = \text{Residue at } -\beta = \frac{\beta(-1)^{n+1}}{(\alpha - \beta)\beta^{n+1}}.$$

Combining the above with the facts that $\alpha + \beta = 1$ and $\alpha\beta = -1$ we deduce (2).

This elegant application of complex analysis [4] seems to have gone unnoticed in the undergraduate textbook literature. For an application to tribonacci sequences, whose generating functions contain cubic denominators, see [11].

In conclusion, we have illustrated, using definitions of the Fibonacci numbers, how a variety of techniques can be used to derive the same theorem and provide identical homework problems in superficially totally different courses. It is advocated that instructors and undergraduate texts find similar mathematical "scenes" for the itinerary of the touring undergraduate.

Note. Written while the author was affiliated with Dowling College.

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The Integral of $x^{1/2}$, etc.

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In calculus we derive $\int_0^x t^2\,dt = (x^3/3)$ by a limiting process involving Riemann sums. Known formulas for summing $\sum_{k=1}^n k^m$, where m is a positive integer, are used in computing the limit. Using a similar technique we show that $\int_0^x t^{1/2}\,dt = (2x^{3/2}/3)$. The method can be used to integrate $f(t) = t^{p/q}$, where p and q are positive integers.

For the integrand $f(t) = t^{1/2}$, the partition of $0 \le t \le x$ is chosen to be

$$\left\{ x_k = \frac{k^2 x}{n^2} \right\}_{k=0}^{k=n}$$

(which involves k^2 because $q = 2$), and the corresponding function values are